COMPANION NOTES
A Working Excursion to Accompany Baby Rudin

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Preface

These notes have been prepared to assist students who are learning Advanced Calculus/Real Analysis for the first time in courses or self-study programs that are using the text *Principles of Mathematical Analysis* (3rd Edition) by Walter Rudin. References to page numbers or general location of results that mention "our text" are always referring to Rudin's book. The notes are designed to

- encourage or engender an interactive approach to learning the material,
- provide more examples at the introductory level,
- offer some alternative views of some of the concepts, and
- draw a clearer connection to the mathematics that is prerequisite to understanding the development of the mathematical analysis.

On our campus, the only prerequisites on the Advanced Calculus course include an introduction to abstract mathematics (MAT108) course and elementary calculus. Consequently, the terseness of Rudin can require quite an intellectual leap. One needs to pause and reflect on what is being presented; stopping to do things like draw pictures, construct examples or counterexamples for the concepts the are being discussed, and learn the definitions is an essential part of learning the material. These *Companion Notes* explicitly guide the reader/participant to engage in those activities. With more math experience or maturity such behaviors should become a natural part of learning mathematics. A math text is not a novel; simply reading it from end to end is unlikely to give you more than a sense for the material. On the other hand, the level of interaction that is needed to successfully internalize an understanding of the material varies widely from person to person. For optimal benefit from the combined use of the text (Rudin) and the *Companion Notes* first read the section of interest as offered in Rudin, then work through the relevant section or sections in the *Companion Notes*, and follow that by a more interactive review of the section from Rudin with which you started.

One thing that should be quite noticeable is the higher level of detail that is offered for many of the proofs. This was done largely in response to our campus prerequisite for the course. Because most students would have had only a brief exposure to some of the foundational material, a very deliberate attempt has been made to demonstrate how the prerequisite material that is usually learned in an introduction to abstract mathematics course is directly applied to the development of mathematical analysis. You always have the elegant, “no nonsense” approach available in the text. Learn to pick and choose the level of detail that you need according to your own personal mathematical needs.

0.0.1 About the Organization of the Material

The chapters and sections of the *Companion Notes* are not identically matched with their counterparts in the text. For example, the material related to Rudin’s Chapter 1 can be found in Chapter 1, Chapter 2 and the beginning of Chapter 3 of the *Companion Notes*. There are also instances of topic coverage that haven’t made it into the *Companion Notes;* the exclusions are due to course timing constraints and not statements concerning importance of the topics.

0.0.2 About the Errors

Of course, there are errors! In spite of my efforts to correct typos and adjust errors as they have been reported to me by my students, I am sure that there are more errors to be found and I hope for the assistance of students who find things that look like errors as they work through the notes. If you encounter errors or things that look like errors, please sent me a brief email indicating the nature of the problem. My email address is emsilvia@math.ucdavis.edu. Thank you in advance for any comments, corrections, and/or insights that you decide to share.

Chapter 1

The Field of Reals and Beyond

Our goal with this section is to develop (review) the basic structure that characterizes the set of real numbers. Much of the material in the first section is a review of properties that were studied in MAT108; however, there are a few slight differences in the definitions for some of the terms. Rather than prove that we can get from the presentation given by the author of our MAT127A textbook to the previous set of properties, with one exception, we will base our discussion and derivations on the new set. As a general rule the definitions offered in this set of *Companion Notes* will be stated in symbolic form; this is done to reinforce the language of mathematics and to give the statements in a form that clarifies how one might prove satisfaction or lack of satisfaction of the properties. YOUR GLOSSARIES ALWAYS SHOULD CONTAIN THE (IN SYMBOLIC FORM) DEFINITION AS GIVEN IN OUR NOTES because that is the form that will be required for successful completion of literacy quizzes and exams where such statements may be requested.

1.1 Fields

Recall the following **DEFINITIONS:**

- The **Cartesian product** of two sets A and B, denoted by $A \times B$, is

$$\{(a, b) : a \in A \wedge b \in B\}.$$

- A **function** h from A into B is a subset of $A \times B$ such that

  (i) $(\forall a)\,[a \in A \Rightarrow (\exists b)\,(b \in B \wedge (a, b) \in h)]$; i.e., $\operatorname{dom} h = A$, and

  (ii) $(\forall a)\,(\forall b)\,(\forall c)\,[(a, b) \in h \wedge (a, c) \in h \Rightarrow b = c]$; i.e., h is single-valued.

- A **binary operation** on a set A is a function from $A \times A$ into A.

- A **field** is an algebraic structure, denoted by $(\mathbb{F}, +, \cdot, e, f)$, that includes a set of objects, $\mathbb{F}$, and two binary operations, addition $(+)$ and multiplication $(\cdot)$, that satisfy the Axioms of Addition, Axioms of Multiplication, and the Distributive Law as described in the following list.

  (A) **Axioms of Addition** ($(\mathbb{F}, +, e)$ is a commutative group under the binary operation of addition $(+)$ with the additive identity denoted by e);

  (A1) $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$

  (A2) $(\forall x)\,(\forall y)\,(x, y \in \mathbb{F} \Rightarrow (x + y = y + x))$ (commutative with respect to addition)

  (A3) $(\forall x)\,(\forall y)\,(\forall z)\,\big(x, y, z \in \mathbb{F} \Rightarrow \big[(x + y) + z = x + (y + z)\big]\big)$ (associative with respect to addition)

  (A4) $(\exists e)\,[e \in \mathbb{F} \wedge (\forall x)\,(x \in \mathbb{F} \Rightarrow x + e = e + x = x)]$ (additive identity property)

  (A5) $(\forall x)\,(x \in \mathbb{F} \Rightarrow (\exists\,(-x))\,[(-x) \in \mathbb{F} \wedge (x + (-x) = (-x) + x = e)])$ (additive inverse property)

  (M) **Axioms of Multiplication** ($(\mathbb{F}, \cdot, f)$ is a commutative group under the binary operation of multiplication $(\cdot)$ with the multiplicative identity denoted by f);

  (M1) $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$

  (M2) $(\forall x)\,(\forall y)\,(x, y \in \mathbb{F} \Rightarrow (x \cdot y = y \cdot x))$ (commutative with respect to multiplication)

  (M3) $(\forall x)\,(\forall y)\,(\forall z)\,\big(x, y, z \in \mathbb{F} \Rightarrow \big[(x \cdot y) \cdot z = x \cdot (y \cdot z)\big]\big)$ (associative with respect to multiplication)

  (M4) $(\exists f)\,\big[f \in \mathbb{F} \wedge f \neq e \wedge (\forall x)\,(x \in \mathbb{F} \Rightarrow x \cdot f = f \cdot x = x)\big]$ (multiplicative identity property)

  (M5) $(\forall x)\,(x \in \mathbb{F} - \{e\} \Rightarrow$
  $$\big[\big(\exists\,\big(x^{-1}\big)\big)\,\big(x^{-1} \in \mathbb{F} \wedge (x \cdot (x^{-1}) = (x^{-1}) \cdot x = f)\big]\big)$$
  (multiplicative inverse property)

(D) **The Distributive Law**

$$(\forall x)\,(\forall y)\,(\forall z)\,(x, y, z \in \mathbb{F} \Rightarrow [x \cdot (y + z) = (x \cdot y) + (x \cdot z)])$$

Remark 1.1.1 *Properties (A1) and (M1) tell us that* $\mathbb{F}$ *is closed under addition and closed under multiplication, respectively.*

Remark 1.1.2 *The additive identity and multiplicative identity properties tell us that a field has at least two elements; namely, two distinct identities. To see that two elements is enough, note that, for* $\mathbb{F} = \{0, 1\}$*, the algebraic structure* $(\mathbb{F}, \oplus, \otimes, 0, 1)$ *where* $\oplus : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ *and* $\otimes : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ *are defined by the following tables:*

$\oplus$	0	1
0	0	1
1	1	0

$\otimes$	0	1
0	0	0
1	0	1

,

is a field.

Remark 1.1.3 *The fields with which you are probably the most comfortable are the rationals* $(\mathbb{Q}, +, \cdot, 0, 1)$ *and the reals* $(\mathbb{R}, +, \cdot, 0, 1)$*. A field that we will discuss shortly is the complex numbers* $(\mathbb{C}, +, \cdot, (0, 0), (1, 0))$ *Since each of these distinctly different sets satisfy the same list of field properties, we will expand our list of properties in search of ones that will give us distinguishing features.*

When discussing fields, we should distinguish that which can be claimed as a basic field property ((A),(M), and (D)) from properties that can (and must) be proved from the basic field properties. For example, given that $(\mathbb{F}, +, \cdot)$ is a field, we can claim that $(\forall x)\,(\forall y)\,(x, y \in \mathbb{F} \Rightarrow x + y \in \mathbb{F})$ as an alternative description of property (A1) while we can not claim that additive inverses are unique. The latter observation is important because it explains why we can't claim $e = w$ from $(\mathbb{F}, +, \cdot, e, f)$ being a field and $x + w = x + e = x$; we don't have anything that allows us to "subtract from both sides of an equation". The relatively small number of properties that are offered in the definition of a field motivates our search for additional properties of fields that can be proved using only the basic field properties and elementary logic. In general, we don't claim as axioms that which can be proved from the "minimal" set of axioms that comprise the definition of a field. We will list some properties that require proof and offer some proofs to illustrate an approach to doing such proofs. A slightly different listing of properties with proofs of the properties is offered in Rudin.

Proposition 1.1.4 *Properties for the Additive Identity of a field* $(\mathbb{F}, +, \cdot, e, f)$

1. $(\forall x)\,(x \in \mathbb{F} \wedge x + x = x \Rightarrow x = e)$
2. $(\forall x)\,(x \in \mathbb{F} \Rightarrow x \cdot e = e \cdot x = e)$
3. $(\forall x)\,(\forall y)\left[(x, y \in \mathbb{F} \wedge x \cdot y = e) \Rightarrow (x = e \vee y = e)\right]$

Proof. (of #1) Suppose that $x \in \mathbb{F}$ satisfies $x + x = x$. Since $x \in \mathbb{F}$, by the additive inverse property, $-x \in \mathbb{F}$ is such that $x + -x = -x + x = e$. Now by substitution and the associativity of addition,

$$e = x + (-x) = (x + x) + (-x) = x + (x + -x) = x + e = x.$$

(of #3) Suppose that $x, y \in \mathbb{F}$ are such that $x \cdot y = e$ and $x \neq e$. Then, by the multiplicative inverse property, $x^{-1} \in F$ satisfies $x \cdot x^{-1} = x^{-1} \cdot x = f$. Then substitution, the associativity of multiplication, and #2 yields that

$$y = f \cdot y = \left(x^{-1} \cdot x\right) \cdot y = x^{-1} \cdot (x \cdot y) = x^{-1} \cdot e = e.$$

Hence, for $x, y \in \mathbb{F}$, $x \cdot y = e \wedge x = e$ implies that $y = e$. The claim now follows immediately upon noting that, for any propositions P, Q, and M, $[P \Rightarrow (Q \vee M)]$ is logically equivalent to $[(P \wedge \neg Q) \Rightarrow M]$. ■

Excursion 1.1.5 *Use #1 to prove #2.*

The key here was to work from $x \cdot e = x\,(e + e)$.

Proposition 1.1.6 *Uniqueness of Identities and Inverses for a field* $(\mathbb{F}, +, \cdot, e, f)$

1. *The additive identity of a field is unique.*

2. *The multiplicative identity of a field is unique.*
3. *The additive inverse of any element in $\mathbb{F}$ is unique.*
4. *The multiplicative inverse of any element in $\mathbb{F} - \{e\}$ is unique.*

Proof. (of #1) Suppose that $w \in \mathbb{F}$ is such that

$$(\forall x)\,(x \in \mathbb{F} \Rightarrow x + w = w + x = x)\,.$$

In particular, since $e \in \mathbb{F}$, we have that $e = e + w$. Since e is given as an additive identity and $w \in \mathbb{F}$, $e + w = w$. From the transitivity of equals, we conclude that $e = w$. Therefore, the additive identity of a field is unique.

(of #3) Suppose that $a \in \mathbb{F}$ is such that there exists $w \in \mathbb{F}$ and $x \in \mathbb{F}$ satisfying

$$a + w = w + a = e \qquad and \qquad a + x = x + a = e.$$

From the additive identity and associative properties

$$\begin{aligned} w = w + e &= w + (a + x) \\ &= (w + a) + x \\ &= e + x \\ &= x. \end{aligned}$$

Since a was arbitrary, we conclude that the additive inverse of each element in a field is unique. ■

Excursion 1.1.7 *Prove #4.*

Completing this excursion required only appropriate modification of the proof that was offered for #3. You needed to remember to take you arbitrary element in F to not be the additive identity and then simply change the operation to multiplication. Hopefully, you remembered to start with one of the inverses of your arbitrary element and work to get it equal to the other one.

Proposition 1.1.8 *Sums and Products Involving Inverses for a field* $(\mathbb{F},+,\cdot,e,\ f)$

1. $(\forall a)\,(\forall b)\,(a,b\in\mathbb{F}\Rightarrow-(a+b)=(-a)+(-b))$
2. $(\forall a)\,(a\in F\Rightarrow-(-a)=a)$
3. $(\forall a)\,(\forall b)\,(a,b\in\mathbb{F}\Rightarrow a\cdot(-b)=-(a\cdot b))$
4. $(\forall a)\,(\forall b)\,(a,b\in\mathbb{F}\Rightarrow(-a)\cdot b=-(a\cdot b))$
5. $(\forall a)\,(\forall b)\,(a,b\in\mathbb{F}\Rightarrow(-a)\cdot(-b)=a\cdot b)$
6. $(\forall a)\left(a\in\mathbb{F}-\{e\}\Rightarrow\left(a^{-1}\neq e\wedge\left[a^{-1}\right]^{-1}=a\wedge-\left(a^{-1}\right)=(-a)^{-1}\right)\right)$
7. $(\forall a)\,(\forall b)\left(a,b\in\mathbb{F}-\{e\}\Rightarrow(a\cdot b)^{-1}=\left(a^{-1}\right)\left(b^{-1}\right)\right)$

Proof. *(of #2) Suppose that* $a\in\mathbb{F}$. *By the additive inverse property* $-a\in\mathbb{F}$ *and* $-(-a)\in\mathbb{F}$ *is the additive inverse of* $-a$*; i.e.,* $-(-a)+(-a)=e$. *Since* $-a$ *is the additive inverse of* a, $(-a)+a=a+(-a)=e$ *which also justifies that* a *is an additive inverse of* $-a$. *From the uniqueness of additive inverses (Proposition 1.1.6), we conclude that* $-(-a)=a$. ■

Excursion 1.1.9 *Fill in what is missing in order to complete the following proof of #6.*

Proof. *Suppose that* $a\in\mathbb{F}-\{e\}$. *From the multiplicative inverse property,* $a^{-1}\in\mathbb{F}$ *satisfies* $\underset{(1)}{\underline{\qquad\qquad\qquad\qquad}}$. *If* $a^{-1}=e$, *then, by Proposition 1.1.4(#2),* $a^{-1}\cdot a=e$. *Since multiplication is single-valued, this would imply that* $\underset{(2)}{\underline{\qquad\qquad}}$ *which contradicts part of the* $\underset{(3)}{\underline{\qquad\qquad\qquad\qquad}}$ *property. Thus,* $a^{-1}\neq e$.

Since $a^{-1}\in\mathbb{F}-\{e\}$, *by the* $\underset{(4)}{\underline{\qquad\qquad\qquad\qquad}}$ *property,* $\left(a^{-1}\right)^{-1}\in\mathbb{F}$ *and satisfies* $\left(a^{-1}\right)^{-1}\cdot a^{-1}=a^{-1}\cdot\left(a^{-1}\right)^{-1}=f$*; but this equation also justifies that* $\left(a^{-1}\right)^{-1}$ *is a multiplicative inverse for* a^{-1}. *From Proposition* $\underset{(5)}{\underline{\qquad\qquad}}$, *we conclude that* $\left(a^{-1}\right)^{-1}=a$.

From (#5), $\left(-\left(a^{-1}\right)\right)\cdot(-a) = a^{-1}\cdot a = f$ *from which we conclude that* $-\left(a^{-1}\right)$ *is a* $\underset{(6)}{\underline{\hspace{6cm}}}$ *for* $-a$*. Since* $(-a)^{-1}$ *is a multiplicative inverse for* $(-a)$ *and multiplicative inverses are unique, we have that* $-\left(a^{-1}\right) = (-a)^{-1}$ *as claimed.* ■

Acceptable responses are: (1) $a\cdot a^{-1} = f$, (2) $e = f$, (3) multiplicative identity, (4) multiplicative inverse, (5) 1.1.6(#4), and (6) multiplicative inverse.

Proposition 1.1.10 *Solutions to Linear Equations. Given a field* $(\mathbb{F}, +, \cdot, 0, 1)$*,*

1. $(\forall a)\,(\forall b)\,(a, b \in \mathbb{F} \Rightarrow (\exists! x)\,(x \in \mathbb{F} \wedge a + x = b))$
2. $(\forall a)\,(\forall b)\,(a, b \in \mathbb{F} \wedge a \neq 0 \Rightarrow (\exists! x)\,(x \in \mathbb{F} \wedge a \cdot x = b))$

Proof. *(of #1) Suppose that* $a, b \in \mathbb{F}$ *and* $a \neq 0$*. Since* $a \in \mathbb{F} - \{0\}$ *there exists* $a^{-1} \in \mathbb{F}$ *such that* $a\cdot a^{-1} = a^{-1}\cdot a = 1$*. Because* $a^{-1} \in \mathbb{F}$ *and* $b \in \mathbb{F}$*,* $x \underset{def}{=} a^{-1}\cdot b \in \mathbb{F}$ *from (M1). Substitution and the associativity of multiplication yield that*

$$a\cdot x = a\cdot\left(a^{-1}\cdot b\right) = \left(a\cdot a^{-1}\right)\cdot b = 1\cdot b = b.$$

Hence, x *satisfies* $a\cdot x = b$*. Now, suppose that* $w \in \mathbb{F}$ *also satisfies* $a\cdot w = b$*. Then*

$$w = 1\cdot w = \left(a^{-1}\cdot a\right)\cdot w = a^{-1}\cdot(a\cdot w) = a^{-1}\cdot b = x.$$

Since a *and* b *were arbitrary,*

$$(\forall a)\,(\forall b)\,(a, b \in \mathbb{F} \Rightarrow (\exists! x)\,(x \in \mathbb{F} \wedge a + x = b))\,.$$

■

Remark 1.1.11 *As a consequence of Proposition 1.1.10, we now can claim that, if* $x, w, z \in \mathbb{F}$ *and* $x + w = x + z$*, then* $w = z$ *and; if* $w, z \in \mathbb{F}$*,* $x \in \mathbb{F} - \{0\}$ *and* $w\cdot x = z\cdot x$*, then* $w = z$*. The justification is the uniqueness of solutions to linear equations in a field. In terms of your previous experience with elementary algebraic manipulations used to solve equations, the proposition justifies what is commonly referred to as "adding a real number to both sides of an equation" and "dividing both sides of an equation by a nonzero real number."*

Proposition 1.1.12 *Addition and Multiplication Over Fields Containing Three or More Elements. Suppose that* $(\mathbb{F},+,\cdot)$ *is a field and* $a,b,c,d\in\mathbb{F}$. *Then*

1. $a+b+c=a+c+b=\cdots=c+b+a$
2. $a\cdot b\cdot c=a\cdot c\cdot b=\cdots=c\cdot b\cdot a$
3. $(a+c)+(b+d)=(a+b)+(c+d)$
4. $(a\cdot c)\cdot(b\cdot d)=(a\cdot b)\cdot(c\cdot d)$

Proposition 1.1.13 *Multiplicative Inverses in a field* $(\mathbb{F},+,\cdot,0,1)$

1. $(\forall a)\,(\forall b)\,(\forall c)\,(\forall d)\left[\begin{array}{l}(a,b,c,d\in\mathbb{F}\wedge b\neq 0\wedge d\neq 0)\\ \Rightarrow b\cdot d\neq 0\wedge\left(a\cdot b^{-1}\right)\cdot\left(c\cdot d^{-1}\right)=(a\cdot c)\cdot(b\cdot d)^{-1}\end{array}\right]$
2. $(\forall a)\,(\forall b)\,(\forall c)\left[(a,b,c\in\mathbb{F}\wedge c\neq 0)\Rightarrow\left(a\cdot c^{-1}\right)+\left(b\cdot c^{-1}\right)=(a+b)\cdot c^{-1}\right]$
3. $(\forall a)\,(\forall b)\,[(a,b\in\mathbb{F}\wedge b\neq 0)\Rightarrow\left((-a)\cdot b^{-1}\right)=\left(a\cdot(-b)^{-1}\right)=-\left(a\cdot b^{-1}\right)]$
4. $(\forall a)\,(\forall b)\,(\forall c)\,(\forall d)\,[(a\in\mathbb{F}\wedge b,c,d\in\mathbb{F}-\{0\})\Rightarrow c\cdot d^{-1}\neq 0$
   $\wedge\left(a\cdot b^{-1}\right)\cdot\left(c\cdot d^{-1}\right)^{-1}=(a\cdot d)\cdot(b\cdot c)^{-1}=\left(a\cdot b^{-1}\right)\cdot\left(d\cdot c^{-1}\right)]$
5. $(\forall a)\,(\forall b)\,(\forall c)\,(\forall d)\,[(a,c\in\mathbb{F}\wedge b,d\in\mathbb{F}-\{0\})\Rightarrow b\cdot d\neq 0\wedge$
   $\left(a\cdot b^{-1}\right)+\left(c\cdot d^{-1}\right)=(a\cdot d+b\cdot c)\cdot(b\cdot d)^{-1}]$

Proof. *(of #3) Suppose* $a,b\in\mathbb{F}$ *and* $b\neq 0$. *Since* $b\neq 0$, *the zero of the field is its own additive inverse, and additive inverses are unique, we have that* $-b\neq 0$. *Since* $a\in\mathbb{F}$ *and* $b\in\mathbb{F}-\{0\}$ *implies that* $-a\in\mathbb{F}$ *and* $b^{-1}\in\mathbb{F}-\{0\}$, *by Proposition 1.1.8(#4),* $(-a)\cdot b^{-1}=-\left(a\cdot b^{-1}\right)$. *From Proposition 1.1.8(#6), we know that* $-\left(b^{-1}\right)=(-b)^{-1}$. *From the distributive law and Proposition 1.1.8(#2),*

$$a\cdot(-b)^{-1}+a\cdot b^{-1}=a\cdot\left((-b)^{-1}+b^{-1}\right)=a\cdot\left(-\left(b^{-1}\right)+b^{-1}\right)=a\cdot 0=0$$

from which we conclude that $a\cdot(-b)^{-1}$ *is an additive inverse for* $a\cdot b^{-1}$. *Since additive inverses are unique, it follows that* $a\cdot(-b)^{-1}=-\left(a\cdot b^{-1}\right)$. *Combining our results yields that*

$$(-a)\cdot b^{-1}=-\left(a\cdot b^{-1}\right)=a\cdot(-b)^{-1}$$

as claimed. ■

Excursion 1.1.14 *Fill in what is missing in order to complete the following proof of #4.*

Proof. *(of #4) Suppose that* $a \in \mathbb{F}$ *and* $b, c, d \in \mathbb{F} - \{0\}$. *Since* $d \in \mathbb{F} - \{0\}$, *by Proposition* $\underset{(1)}{\underline{\qquad\quad}}$, $d^{-1} \neq 0$. *From the contrapositive of Proposition 1.1.4(#3),* $c \neq 0$ *and* $d^{-1} \neq 0$ *implies that* $\underset{(2)}{\underline{\qquad\qquad\qquad}}$. *In the following, the justifications for the step taken is provided on the line segment to the right of the change that has been made.*

$$
\begin{aligned}
\left(a \cdot b^{-1}\right) \cdot \left(c \cdot d^{-1}\right)^{-1} &= \left(a \cdot b^{-1}\right) \cdot \left(c^{-1} \cdot \left(d^{-1}\right)^{-1}\right) && \underset{(3)}{\underline{\qquad\qquad\qquad}} \\
&= \left(a \cdot b^{-1}\right) \cdot \left(c^{-1} \cdot d\right) && \underset{(4)}{\underline{\qquad\qquad\qquad}} \\
&= a \cdot \left(b^{-1} \cdot \left(c^{-1} \cdot d\right)\right) && \underset{(5)}{\underline{\qquad\qquad\qquad}} \\
&= a \cdot \left(\left(b^{-1} \cdot c^{-1}\right) \cdot d\right) && \underset{(6)}{\underline{\qquad\qquad\qquad}} \\
&= a \cdot \left(d \cdot \left(b^{-1} \cdot c^{-1}\right)\right) && \underset{(7)}{\underline{\qquad\qquad\qquad}} \\
&= a \cdot \left(d \cdot (b \cdot c)^{-1}\right) && \underset{(8)}{\underline{\qquad\qquad\qquad}} \\
&= (a \cdot d) \cdot (b \cdot c)^{-1}. && \underset{(9)}{\underline{\qquad\qquad\qquad}}
\end{aligned}
$$

From Proposition 1.1.8(#7) combined with the associative and commutative properties of addition we also have that

$$
\begin{aligned}
(a \cdot d) \cdot (b \cdot c)^{-1} &= (a \cdot d) \cdot \left(b^{-1} \cdot c^{-1}\right) \\
&= \left((a \cdot d) \cdot b^{-1}\right) \cdot c^{-1} \\
&= \left(a \cdot \left(d \cdot b^{-1}\right)\right) \cdot c^{-1} \\
&= \underset{(10)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}} \\
&= \left(\left(a \cdot b^{-1}\right) \cdot d\right) \cdot c^{-1} \\
&= \left(a \cdot b^{-1}\right) \cdot \left(d \cdot c^{-1}\right).
\end{aligned}
$$

Consequently, $\left(a \cdot b^{-1}\right) \cdot \left(c \cdot d^{-1}\right)^{-1} = (a \cdot d) \cdot (b \cdot c)^{-1} = \left(a \cdot b^{-1}\right) \cdot \left(d \cdot c^{-1}\right)$ *as claimed.* ■

Acceptable responses are: (1) 1.1.8(#6), (2) $c \cdot d^{-1} \neq 0$, (3) Proposition 1.1.8(#7), (4) Proposition 1.1.8(#6), (5) associativity of multiplication, (6) associativity of multiplication, (7) commutativity of multiplication, (8) Proposition 1.1.8(#7), (9) associativity of multiplication, (10) $\left(a \cdot \left(b^{-1} \cdot d\right)\right) \cdot c^{-1}$.

The list of properties given in the propositions is, by no means, exhaustive. The propositions illustrate the kinds of things that can be concluded (proved) from the core set of basic field axioms.

Notation 1.1.15 *We have listed the properties without making use of some* ***notational conventions*** *that can make things look simpler. The two that you might find particularly helpful are that*

- the expression $a + (-b)$ may be written as $a - b$; $(-a) + (-b)$ may be written as $-a - b$; and
- the expression $a \cdot b^{-1}$ may be written as $\frac{a}{b}$. (Note that applying this notational convention to the Properties of Multiplicative Inverses stated in the last proposition can make it easier for you to remember those properties.)

Excursion 1.1.16 *On the line segments provided, fill in appropriate justifications for the steps given in the following outline of a proof that for* a, b, c, d *in a field,* $(a + b) - (c - d) = (a - c) + (b + d)$.

Observation		*Justification*
$(a + b) - (c - d) = (a + b) + (-(c + (-d)))$		*notational convention*
$(a + b) + (-(c + (-d))) = (a + b) + ((-c) + (-(-d)))$		(1)
$(a + b) + ((-c) + (-(-d))) = (a + b) + ((-c) + d)$		(2)
$(a + b) + ((-c) + d) = a + (b + ((-c) + d))$		(3)
$a + (b + ((-c) + d)) = a + ((b + (-c)) + d)$		(4)
$a + ((b + (-c)) + d) = a + (((-c) + b) + d)$		(5)
$a + (((-c) + b) + d) = a + ((-c) + (b + d))$		(6)
$a + ((-c) + (b + d)) = (a + (-c)) + (b + d)$		(7)
$(a + (-c)) + (b + d) = (a - c) + (b + d)$		(8)

*******Acceptable responses are: (1) Proposition 1.1.8(#1), (2) Proposition 1.1.8(#2), (3) and (4) associativity of addition, (5) commutativity of addition, (6) and (7) associativity of addition, and (8) notational convention.*******

1.2 Ordered Fields

Our basic field properties and their consequences tell us how the binary operations function and interact. The set of basic field properties doesn't give us any means of comparison of elements; more structure is needed in order to formalize ideas such as "positive elements in a field" or "listing elements in a field in increasing order." To do this we will introduce the concept of an ordered field.

Recall that, for any set S, a **relation on** S is any subset of $S \times S$

Definition 1.2.1 *An* ***order****, denoted by* $<$, ***on a set*** S *is a relation on* S *that satisfies the following two properties:*

1. ***The Trichotomy Law****: If* $x \in S$ *and* $y \in S$*, then one and only one of*

$$(x < y) \text{ or } (x = y) \text{ or } (y < x)$$

is true.

2. ***The Transitive Law****:* $(\forall x)\,(\forall y)\,(\forall z)\left[x, y, z \in S \wedge x < y \wedge y < z \Rightarrow x < z\right]$.

Remark 1.2.2 *Satisfaction of the Trichotomy Law requires that*

$$(\forall x)\,(\forall y)\,(x, y \in S \Rightarrow (x = y) \vee (x < y) \vee (y < x))$$

be true and that each of

$$\begin{aligned}&(\forall x)\,(\forall y)\,(x, y \in S \Rightarrow ((x = y) \Rightarrow \neg(x < y) \wedge \neg(y < x))),\\&(\forall x)\,(\forall y)\,(x, y \in S \Rightarrow ((x < y) \Rightarrow \neg(x = y) \wedge \neg(y < x))), \text{ and}\\&(\forall x)\,(\forall y)\,(x, y \in S \Rightarrow ((y < x) \Rightarrow \neg(x = y) \wedge \neg(x < y)))\end{aligned}$$

be true. The first statement, $(\forall x)\,(\forall y)\,(x, y \in S \Rightarrow (x = y) \vee (x < y) \vee (y < x))$ *is not equivalent to the Trichotomy Law because the disjunction is not mutually exclusive.*

Example 1.2.3 *For* $S = \{a, b, c\}$ *with* a, b, *and* c *distinct,* $< = \{(a, b), (b, c), (a, c)\}$ *is an order on* S. *The notational convention for* $(a, b) \in <$ *is* $a < b$. *The given ordering has the minimum and maximum number of ordered pairs that is needed to meet the definition. This is because, given any two distinct elements of* S, x *and* y, *we must have one and only one of* $(x, y) \in <$ *or* $(y, x) \in <$. *After making free choices of two ordered pairs to go into an acceptable ordering for* S, *the choice of the third ordered pair for inclusion will be determined by the need to have the Transitive Law satisfied.*

Remark 1.2.4 *The definition of a particular order on a set* S *is, to a point, up to the definer. You can choose elements of* $S \times S$ *almost by preference until you start having enough elements to force the choice of additional ordered pairs in order to meet the required properties. In practice, orders are defined by some kind of formula or equation.*

Example 1.2.5 *For* $\mathbb{Q}$, *the set of rationals, let* $<\subset \mathbb{Q} \times \mathbb{Q}$ *be defined by* $(r, s) \in <\Leftrightarrow (s + (-r))$ *is a positive rational. Then* $(\mathbb{Q}, <)$ *is an ordered set.*

Remark 1.2.6 *The treatment of ordered sets that you saw in MAT108 derived the Trichotomy Law from a set of properties that defined a linear order on a set. Given an order* $<$ *on a set, we write* $x \leq y$ *for* $(x < y) \vee x = y$. *With this notation, the two linear ordering properties that could have been introduced and used to prove the Trichotomy Law are the Antisymmetric law,*

$$(\forall x)\,(\forall y)\,((x, y \in S \wedge (x, y) \in \leq \wedge (y, x) \in \leq) \Rightarrow x = y),$$

and the Comparability Law,

$$(\forall x)\,(\forall y)\,(x, y \in S \Rightarrow ((x, y) \in \leq \vee (y, x) \in \leq)).$$

Now, because we have made satisfaction of the Trichotomy Law part of the definition of an order on a set, we can claim that the Antisymmetric Law and the Comparability Law are satisfied for an ordered set.

Definition 1.2.7 *An* ***ordered field*** $(\mathbb{F}, +, \cdot, 0, 1, <)$ *is an ordered set that satisfies the following two properties.*

(OF1) $(\forall x)\,(\forall y)\,(\forall z)\left[x, y, z \in \mathbb{F} \wedge x < y \Rightarrow x + z < y + z\right]$

(OF2) $(\forall x)\,(\forall y)\,(\forall z)\left[x, y, z \in \mathbb{F} \wedge x < y \wedge 0 < z \Rightarrow x \cdot z < y \cdot z\right]$

Remark 1.2.8 *In the definition of ordered field offered here, we have deviated from one of the statements that is given in our text. The second condition given in the text is that*

$$(\forall x)\,(\forall y)\left[x, y \in \mathbb{F} \wedge x > 0 \wedge y > 0 \Rightarrow x \cdot y > 0\right];$$

let's denote this proposition by $(altOF2)$. *We will show that satisfaction of* $(OF1)$ *and* $(altOF2)$ *is, in fact, equivalent to satisfaction of* $(OF1)$ *and* $(OF2)$. *Suppose that* $(OF1)$ *and* $(OF2)$ *are satisfied and let* $x, y \in \mathbb{F}$ *be such that* $0 < x$ *and* $0 < y$. *From* $(OF2)$ *and Proposition 1.1.4(#2),* $0 = 0 \cdot y < x \cdot y$. *Since x and y were arbitrary, we conclude that* $(\forall x)\,(\forall y)\left[x, y \in \mathbb{F} \wedge x > 0 \wedge y > 0 \Rightarrow x \cdot y > 0\right]$. *Hence,* $(OF2) \Rightarrow (altOF2)$ *from which we have that* $(OF1) \wedge (OF2) \Rightarrow (OF1) \wedge (altOF2)$. *Suppose that* $(OF1)$ *and* $(altOF2)$ *are satisfied and let* $x, y, z \in \mathbb{F}$ *be such that* $x < y$ *and* $0 < z$. *From the additive inverse property* $(-x) \in \mathbb{F}$ *is such that* $[x + (-x) = (-x) + x = 0]$. *From* $(OF1)$ *we have that*

$$0 = x + (-x) < y + (-x).$$

From $(altOF2)$, *the Distributive Law and Proposition 1.1.8 (#4),* $0 < y + (-x)$ *and* $0 < z$ *implies that*

$$0 < (y + (-x)) \cdot z = (y \cdot z) + ((-x) \cdot z) = (y \cdot z) + (-(x \cdot z)).$$

Because $+$ *and* $\cdot$ *are binary operations on* $\mathbb{F}$, $x \cdot z \in \mathbb{F}$ *and* $(y \cdot z) + (-(x \cdot z)) \in \mathbb{F}$. *It now follows from* $(OF1)$ *and the associative property of addition that*

$$0 + x \cdot z < ((y \cdot z) + (-(x \cdot z))) + x \cdot z = (y \cdot z) + (-(x \cdot z) + x \cdot z) = y \cdot z + 0.$$

Hence, $x \cdot z < y \cdot z$. *Since x, y, and z were arbitrary, we have shown that*

$$(\forall x)\,(\forall y)\,(\forall z)\left[x, y, z \in \mathbb{F} \wedge x < y \wedge 0 < z \Rightarrow x \cdot z < y \cdot z\right]$$

which is $(OF2)$. *Therefore,* $(OF1) \wedge (altOF2) \Rightarrow (OF1) \wedge (OF2)$. *Combining the implications yields that*

$$(OF1) \wedge (OF2) \Leftrightarrow (OF1) \wedge (altOF2) \text{ as claimed.}$$

To get from the requirements for a field to the requirements for an ordered field we added a binary relation (a description of how the elements of the field are ordered or comparable) and four properties that describe how the order and the binary operations “interact.” The following proposition offers a short list of other order properties that follow from the basic set.

Proposition 1.2.9 *Comparison Properties Over Ordered Fields.*

For an ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$ *we have each of the following.*

1. $0 < 1$
2. $(\forall x)\,(\forall y)\left[x, y \in \mathbb{F} \wedge x > 0 \wedge y > 0 \Rightarrow x \cdot y > 0\right]$
3. $(\forall x)\,[x \in \mathbb{F} \wedge x > 0 \Rightarrow (-x) < 0]$
4. $(\forall x)\,(\forall y)\left[x, y \in \mathbb{F} \wedge x < y \Rightarrow -y < -x\right]$
5. $(\forall x)\,(\forall y)\,(\forall z)\left[x, y, z \in \mathbb{F} \wedge x < y \wedge z < 0 \Rightarrow x \cdot z > y \cdot z\right]$
6. $(\forall x)\left[x \in \mathbb{F} \wedge x \neq 0 \Rightarrow x \cdot x = x^2 > 0\right]$
7. $(\forall x)\,(\forall y)\left[x, y \in \mathbb{F} \wedge 0 < x < y \Rightarrow 0 < y^{-1} < x^{-1}\right]$

In the Remark 1.2.8, we proved the second claim. We will prove two others. Proofs for all but two of the statements are given in our text.

Proof. (or #1) By the Trichotomy Law one and only one of $0 < 1$, $0 = 1$, or $1 < 0$ is true in the field. From the multiplicative identity property, $0 \neq 1$; thus, we have one and only one of $0 < 1$ or $1 < 0$. Suppose that $1 < 0$. From $OF1$, we have that $0 = 1 + (-1) < 0 + (-1) = -1$; i.e., $0 < -1$. Hence, $OF2$ implies that $(1) \cdot (-1) < (0) \cdot (-1)$ which, by Proposition 1.1.8(#3), is equivalent to $-1 < 0$. But, from the transitivity property, $0 < -1 \wedge -1 < 0 \Rightarrow 0 < 0$ which is a contradiction. ■

Excursion 1.2.10 *Fill in what is missing in order to complete the following proof of Proposition 1.2.9(#4).*

Proof. *Suppose that* $x, y \in \mathbb{F}$ *are such that* $x < y$. *In view of the additive inverse property,* $-x \in \mathbb{F}$ *and* $-y \in \mathbb{F}$ *satisfy*

$$-x + x = x + -x = 0 \qquad \textit{and} \qquad \underset{(1)}{\underline{\hspace{6cm}}}.$$

From $\underset{(2)}{\underline{\hspace{3cm}}}$, $0 = x + -x < y + -x$*; i.e.,* $\underset{(3)}{\underline{\hspace{6cm}}}$

and $0 + -y < \left(\underset{(4)}{\underline{\hspace{3cm}}}\right) + -y$. *Repeated use of commutativity and associativity allows us to conclude that* $(y + -x) + -y = -x$. *Hence* $-y < -x$ *as claimed.* ■

Acceptable responses are: (1) $-y+y=y+-y=0$, (2) OF1, (3) $0<y+-x$, (4) $y+-x$.

Remark 1.2.11 *From Proposition 1.2.9(#1) we see that the two additional properties needed to get from an ordered set to an ordered field led to the requirement that* $(0,1)$ *be an element of the ordering (binary relation). From* $0<1$ *and* $(OF1)$, *we also have that* $1<1+1=2$, $2<2+1=3$*; etc. Using the convention* $\underbrace{1+1+1\cdots+1}_{n\,of\,them}=n$, *the general statement becomes* $0<n<n+1$.

1.2.1 Special Subsets of an Ordered Field

There are three special subsets of any ordered field that are isolated for special consideration. We offer their formal definitions here for completeness and perspective.

Definition 1.2.12 *Let* $(\mathbb{F},+,\cdot,0,1,\leq)$ *be an ordered field. A subset* S *of* $\mathbb{F}$ *is said to be* ***inductive*** *if and only if*

1. $1\in S$ *and*
2. $(\forall x)\,(x\in S\Rightarrow x+1\in S)$.

Definition 1.2.13 *For* $(\mathbb{F},+,\cdot,0,1,\leq)$ *an ordered field, define*

$$\mathbb{N}_{\mathbb{F}}=\bigcap_{S\in\mathfrak{S}}S$$

where $\mathfrak{S}=\{S\subseteq\mathbb{F}:S \text{ is inductive}\}$. *We will call* $\mathbb{N}_{\mathbb{F}}$ *the set of* ***natural numbers*** *of* $\mathbb{F}$.

Note that, $T=\{x\in\mathbb{F}:x\geq 1\}$ is inductive because $1\in T$ and closure of $\mathbb{F}$ under addition yields that $x+1\in\mathbb{F}$ whenever $x\in\mathbb{F}$. Because $(\forall u)\,(u<1\Rightarrow u\notin T)$ and $T\in\mathfrak{S}$, we immediately have that any $n\in\mathbb{N}_{\mathbb{F}}$ satisfies $n\geq 1$.

Definition 1.2.14 *Let* $(\mathbb{F},+,\cdot,0,1,\leq)$ *be an ordered field. The set of* ***integers of*** $\mathbb{F}$, *denoted* $\mathbb{Z}_{\mathbb{F}}$, *is*

$$\mathbb{Z}_{\mathbb{F}}=\{a\in\mathbb{F}:a\in\mathbb{N}_{\mathbb{F}}\vee\ -a\in\mathbb{N}_{\mathbb{F}}\vee a=0\}\,.$$

It can be proved that both the natural numbers of a field and the integers of a field are closed under addition and multiplication. That is,

$$(\forall m)\,(\forall n)\,(n \in \mathbb{N}_{\mathbb{F}} \wedge m \in \mathbb{N}_{\mathbb{F}} \Rightarrow n + m \in \mathbb{N}_{\mathbb{F}} \wedge n \cdot m \in \mathbb{N}_{\mathbb{F}})$$

and

$$(\forall m)\,(\forall n)\,(n \in \mathbb{Z}_{\mathbb{F}} \wedge m \in \mathbb{Z}_{\mathbb{F}} \Rightarrow n + m \in \mathbb{Z}_{\mathbb{F}} \wedge n \cdot m \in \mathbb{Z}_{\mathbb{F}})\,.$$

This claim requires proof because the fact that addition and multiplication are binary operations on $\mathbb{F}$ only places $n+m$ and $n \cdot m$ in $\mathbb{F}$ because $\mathbb{N}_{\mathbb{F}} \subset \mathbb{F}$ and $\mathbb{Z}_{\mathbb{F}} \subset \mathbb{F}$.

Proofs of the closure of $\mathbb{N}_{\mathbb{F}} = \mathbb{N}$ under addition and multiplication that you might have seen in MAT108 made use of the Principle of Mathematical Induction. This is a useful tool for proving statements involving the natural numbers.

PRINCIPLE OF MATHEMATICAL INDUCTION (PMI). If S is an inductive set of natural numbers, then $S = \mathbb{N}$.

In MAT108, you should have had lots of practice using the Principle of Mathematical Induction to prove statements involving the natural numbers. Recall that to do this, you start the proof by defining a set S to be the set of natural numbers for which a given statement is true. Once we show that $1 \in S$ and $(\forall k)\,(k \in S \Rightarrow (k+1) \in S)$, we observe that S is an inductive set of natural numbers. Then we conclude, by the Principle of Mathematical Induction, that $S = \mathbb{N}$ which yields that the given statement is true for all $\mathbb{N}$.

Two other principles that are logically equivalent to the Principle of Mathematical Induction and still useful for some of the results that we will be proving in this course are the Well-Ordering Principle and the Principle of Complete Induction:

WELL-ORDERING PRINCIPLE (WOP). Any nonempty set S of natural numbers contains a smallest element.

PRINCIPLE OF COMPLETE INDUCTION (PCI). Suppose S is a nonempty set of natural numbers. If

$$((\forall m)\,(m \in \mathbb{N} \wedge \{k \in \mathbb{N} : k < m\} \subset S) \Rightarrow m \in S)$$

then $S = \mathbb{N}$.

Definition 1.2.15 *Let* $(\mathbb{F},+,\cdot,0,1,\leq)$ *be an ordered field. Define*

$$\mathbb{Q}_{\mathbb{F}}=\left\{r\in\mathbb{F}:(\exists m)\,(\exists n)\left(m,n\in\mathbb{Z}_{\mathbb{F}}\wedge n\neq 0\wedge r=mn^{-1}\right)\right\}.$$

The set $\mathbb{Q}_{\mathbb{F}}$ *is called the set of* ***rational numbers of*** $\mathbb{F}$.

Properties #1 and #5 from Proposition 1.1.13 can be used to show the set of rationals of a field is also closed under both addition and multiplication.

The set of real numbers $\mathbb{R}$ is the ordered field with which you are most familiar. Theorem 1.19 in our text asserts that $\mathbb{R}$ is an ordered field; the proof is given in an appendix to the first chapter. The notation (and numerals) for the corresponding special subsets of $\mathbb{R}$ are:

$$\mathbb{N}=\mathbb{J}=\{1,2,3,4,5,...\}\text{ the set of natural numbers}$$
$$\mathbb{Z}=\{m:(m\in\mathbb{N})\vee(m=0)\vee(-m\in\mathbb{N})\}=\{...,-3,-2,-1,0,1,2,3,...\}$$
$$\mathbb{Q}=\{p\cdot q^{-1}=\tfrac{p}{q}:p,q\in\mathbb{Z}\wedge q\neq 0\}.$$

Remark 1.2.16 *The set of natural numbers may also be referred to as the set of positive integers, while the set of nonnegative integers is* $\mathbb{J}\cup\{0\}$. *Another common term for* $\mathbb{J}\cup\{0\}$ *is the set of whole numbers which may be denoted by* $\mathbb{W}$. *In MAT108, the letter* $\mathbb{N}$ *was used to denote the set of natural numbers, while the author of our MAT127 text is using the letter* J. *To make it clearer that we are referring to special sets of numbers, we will use the "blackboard bold" form of the capital letter. Feel free to use either (the old)* $\mathbb{N}$ *or (the new)* $\mathbb{J}$ *for the natural numbers in the field of reals.*

While $\mathbb{N}$ and $\mathbb{Z}$ are not fields, both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields that have several distinguishing characteristics we will be discussing shortly. Since $\mathbb{Q}\subset\mathbb{R}$ and $\mathbb{R}-\mathbb{Q}\neq\emptyset$, it is natural to want a notation for the set of elements of $\mathbb{R}$ that are not rational. Towards that end, we let $\mathbb{I}rr\underset{def}{=}\mathbb{R}-\mathbb{Q}$ denote the set of irrationals. It was shown in MAT108 that $\sqrt{2}$ is irrational. Because $\sqrt{2}+\left(-\sqrt{2}\right)=0\notin\mathbb{I}rr$ and $\sqrt{2}\cdot\sqrt{2}=2\notin\mathbb{I}rr$, we see that $\mathbb{I}rr$ is not closed under either addition or multiplication.

1.2.2 Bounding Properties

Because both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields we note that "satisfaction of the set of ordered field axioms" is not enough to characterize the set of reals. This naturally prompts us to look for other properties that will distinguish the two algebraic systems. The distinction that we will illustrate in this section is that the set of rationals has "certain gaps." During this (motivational) part of the discussion, you might find it intuitively helpful to visualize the "old numberline" representation for the reals. Given two rationals r and s such that $r < s$, it can be shown that $m = (r + s) \cdot 2^{-1} \in \mathbb{Q}$ is such that $r < m < s$. Then $r_1 = (r + m) \cdot 2^{-1} \in \mathbb{Q}$ and $s_1 = (m + s) \cdot 2^{-1} \in \mathbb{Q}$ are such that $r < r_1 < m$ and $m < s_1 < s$. Continuing this process indefinitely and "marking the new rationals on an imagined numberline" might entice us into thinking that we can "fill in most of the points on the number line between r and s." A rigorous study of the situation will lead us to conclude that the thought is shockingly inaccurate. We certainly know that not all the reals can be found this way because, for example, $\sqrt{2}$ could never be written in the form of $(r + s) \cdot 2^{-1}$ for $r, s \in \mathbb{Q}$. The following excursion will motivate the property that we want to isolate in our formal discussion of bounded sets.

Excursion 1.2.17 *Let* $A = \left\{p \in \mathbb{Q} : p > 0 \wedge p^2 < 2\right\}$ *and* $B = \left\{p \in \mathbb{Q} : p > 0 \wedge p^2 > 2\right\}$. *Now we will expand a bit on the approach used in our text to show that* A *has no largest element and* B *has not smallest element. For* p *a positive rational, let*

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}.$$

Then

$$q^2 - 2 = \frac{2\left(p^2 - 2\right)}{(p + 2)^2}.$$

(a) For $p \in A$, *justify that* $q > p$ *and* $q \in A$.

(b) For $p \in B$, *justify that* $q < p$ *and* $q \in B$.

*******Hopefully you took a few moments to find some elements of A and B in order to get a feel for the nature of the two sets. Finding a q that corresponds to a $p \in A$ and a $p \in B$ would pretty much tell you why the claims are true. For (a), you should have noted that $q > p$ because $\left(p^2-2\right)(p+2)^{-1} < 0$ whenever $p^2 < 2$; then $-\left(p^2-2\right)(p+2)^{-1} > 0$ implies that $q = p + \left(-\left(p^2-2\right)(p+2)^{-1}\right) > p + 0 = p$. That q is rational follows from the fact that the rationals are closed under multiplication and addition. Finally $q^2 - 2 = 2\left(p^2-2\right)(p+2)^{-2} < 0$ yields that $q \in A$ as claimed. For (b), the same reasons extend to the discussion needed here; the only change is that, for $p \in B$, $p^2 > 2$ implies that $\left(p^2-2\right)(p+2)^{-1} > 0$ from which it follows that $-\left(p^2-2\right)(p+2)^{-1} < 0$ and $q = p + \left(-\left(p^2-2\right)(p+2)^{-1}\right) < p + 0 = p$.*******

Now we formalize the terminology that describes the property that our example is intended to illustrate. Let $(S, \leq)$ be an ordered set; i.e., $<$ is an order on the set S. A subset A of S is said to be **bounded above** in S if

$$(\exists u)\,(u \in S \wedge (\forall a)\,(a \in A \Rightarrow a \leq u))\,.$$

Any element $u \in S$ satisfying this property is called an **upper bound** of A in S.

Definition 1.2.18 *Let $(S, \leq)$ be an ordered set. For $A \subset S$, u is a* ***least upper bound*** *or* ***supremum*** *of A in S if and only if*

1. $(u \in S \wedge (\forall a)\,(a \in A \Rightarrow a \leq u))$ *and*
2. $(\forall b)\,[(b \in S \wedge (\forall a)\,(a \in A \Rightarrow a \leq b)) \Rightarrow u \leq b]$.

Notation 1.2.19 *For $(S, \leq)$ an ordered set and $A \subset S$, the* ***least upper bound*** *of A is denoted by* $\operatorname{lub}(A)$ *or* $\sup(A)$.

Since a given set can be a subset of several ordered sets, it is often the case that we are simply asked to find the least upper bound of a given set without specifying the "parent ordered set." When asked to do this, simply find, if it exists, the u that satisfies

$$(\forall a)\,(a \in A \Rightarrow a \leq u) \qquad and \qquad (\forall b)\,[(\forall a)\,(a \in A \Rightarrow a \leq b) \Rightarrow u \leq b]\,.$$

The next few examples illustrate how we can use basic "pre-advanced calculus" knowledge to find some least upper bounds of subsets of the reals.

Example 1.2.20 *Find the* lub $\left\{\dfrac{x}{1+x^2} : x \in \mathbb{R}\right\}$.

From Proposition 1.2.9(#5), we know that, for $x \in \mathbb{R}$, $(1-x)^2 \geq 0$*; this is equivalent to*

$$1 + x^2 \geq 2x$$

from which we conclude that $(\forall x)\left(x \in \mathbb{R} \Rightarrow \dfrac{x}{1+x^2} \leq \dfrac{1}{2}\right)$. *Thus,* $\dfrac{1}{2}$ *is an upper bound for* $\left\{\dfrac{x}{1+x^2} : x \in \mathbb{R}\right\}$. *Since* $\dfrac{1}{1+1^2} = \dfrac{1}{2}$, *it follows that*

$$\text{lub}\left\{\frac{x}{1+x^2} : x \in \mathbb{R}\right\} = \frac{1}{2}.$$

The way that this example was done and presented is an excellent illustration of the difference between scratch work (Phase II) and presentation of an argument (Phase III) in the mathematical process. From calculus (MAT21A or its equivalent) we can show that $f(x) = \dfrac{x}{1+x^2}$ has a relative minimum at $x = -1$ and a relative maximum at $x = 1$; we also know that $y = 0$ is a horizontal asymptote for the graph. Armed with the information that $\left(1, \dfrac{1}{2}\right)$ is a maximum for f, we know that all we need to do is use inequalities to show that $\dfrac{x}{1+x^2} \leq \dfrac{1}{2}$. In the scratch work phase, we can work backwards from this inequality to try to find something that we can claim from what we have done thus far; simple algebra gets use from $\dfrac{x}{1+x^2} \leq \dfrac{1}{2}$ to $1 - 2x + x^2 \geq 0$. Once we see that desire to claim $(1-x)^2 \geq 0$, we are home free because that property is given in one of our propositions about ordered fields.

Excursion 1.2.21 *Find the* lub (A) *for each of the following. Since your goal is simply to find the least upper bound, you can use any pre-advanced calculus information that is helpful.*

1. $A = \left\{ \frac{3 + (-1)^n}{2^{n+1}} : n \in \mathbb{J} \right\}$

2. $A = \{(\sin x)(\cos x) : x \in \mathbb{R}\}$

For (1), let $x_n = \frac{3 + (-1)^n}{2^{n+1}}$; then $x_{2j} = \frac{1}{2^{2j-1}}$ is a sequence that is strictly decreasing from $\frac{1}{2}$ to 0; while x_{2j-1} is also decreasing from $\frac{1}{2}$ to 0. Consequently the terms in A are never greater than $\frac{1}{2}$ with the value of $\frac{1}{2}$ being achieved when $n = 1$ and the terms get arbitrarily close to 0 as n approaches infinity. Hence, $\operatorname{lub}(A) = \frac{1}{2}$. For (2), it is helpful to recall that $\sin x \cos x = \frac{1}{2} \sin 2x$. The well known behavior of the sine function immediately yields that $\operatorname{lub}(A) = \frac{1}{2}$.

Example 1.2.22 *Find* $\operatorname{lub}(A)$ *where* $A = \left\{x \in \mathbb{R} : x^2 + x < 3\right\}$.

What we are looking for here is $\sup(A)$ *where* $A = f^{-1}((-\infty, 3))$ *for* $f(x) = x^2 + x$. *Because*

$$y = x^2 + x \Leftrightarrow y + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2,$$

f *is a parabola with vertex* $\left(-\frac{1}{2}, -\frac{1}{4}\right)$. *Hence,*

$$A = f^{-1}((-\infty, 3)) = \left\{x \in R : \frac{-1 - \sqrt{13}}{2} < x < \frac{-1 + \sqrt{13}}{2}\right\}$$

from which we conclude that $\sup(A) = \frac{-1 + \sqrt{13}}{2}$.

Note that the set $A = \left\{p \in \mathbb{Q} : p > 0 \wedge p^2 < 2\right\}$ is a subset of $\mathbb{Q}$ and a subset of $\mathbb{R}$. We have that $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ are ordered sets where $<$ is defined by $r < s \Leftrightarrow (s + (-r))$ is positive. Now $\operatorname{lub}(A) = \sqrt{2} \notin \mathbb{Q}$; hence, there is no least upper bound of A **in** $S = \mathbb{Q}$, but $A \subset S = \mathbb{R}$ has a least upper bound in $S = \mathbb{R}$. This tells us that the "parent set" is important, gives us a distinction between $\mathbb{Q}$ and $\mathbb{R}$ as ordered fields, and motivates us to name the important distinguishing property.

Definition 1.2.23 *An ordered set $(S, <)$ has the* ***least upper bound property*** *if and only if*

$$(\forall E)\left[\begin{array}{c}(E \subset S \wedge E \neq \emptyset \wedge (\exists \beta)\,(\beta \in S \wedge (\forall a)\,(a \in E \Rightarrow a \leq \beta)))\\ \Rightarrow ((\exists u)\,(u = \operatorname{lub}(E) \wedge u \in S))\end{array}\right]$$

Remark 1.2.24 *As noted above, $(\mathbb{Q}, \leq)$ does not satisfy the* "lub *property", while $(\mathbb{R}, \leq)$ does satisfy this property.*

The proof of the following lemma is left an exercise.

Lemma 1.2.25 *Let $(X, \leq)$ be an ordered set and $A \subseteq X$. If A has a least upper bound in X, it is unique.*

We have analogous or companion definitions for subsets of an ordered set that are bounded below. Let $(S, \leq)$ be an ordered set; i.e., $<$ is an order on the set S. A subset A of S is said to be **bounded below** in S if

$$(\exists v)\,(v \in S \wedge (\forall a)\,(a \in A \Rightarrow v \leq a))\,.$$

Any element $u \in S$ satisfying this property is called a **lower bound** of A in S.

Definition 1.2.26 *Let $(S, \leq)$ be a linearly ordered set. A subset A of S is said to have a* ***greatest lower bound*** *or* ***infimum*** *in S if*

1. $(\exists g)\,(g \in S \wedge (\forall a)\,(a \in A \Rightarrow g \leq a))$, *and*
2. $(\forall c)\left[(c \in S \wedge (\forall a)\,(a \in A \Rightarrow c \leq a)) \Rightarrow c \leq g\right]$.

Example 1.2.27 *Find the* $\operatorname{glb}(A)$ *where* $A = \left\{(-1)^n\left(\dfrac{1}{4} - \dfrac{2}{n}\right) : n \in \mathbb{N}\right\}$.

Let $x_n = (-1)^n\left(\dfrac{1}{4} - \dfrac{2}{n}\right)$; then, for n odd, $x_n = \dfrac{2}{n} - \dfrac{1}{4}$ and, for n even, $x_n = \dfrac{1}{4} - \dfrac{2}{n}$.

Suppose that $n \geq 4$. *By Proposition 1.2.9(#7), it follows that* $\frac{1}{n} \leq \frac{1}{4}$. *Then* $(OF2)$ *and* $(OF1)$ *yield that* $\frac{2}{n} \leq \frac{2}{4} = \frac{1}{2}$ *and* $\frac{2}{n} - \frac{1}{4} \leq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, *respectively. From* $\frac{2}{n} \leq \frac{1}{2}$ *and Proposition 1.2.9(#4), we have that* $-\frac{2}{n} \geq -\frac{1}{2}$. *Thus,* $\frac{1}{4} - \frac{2}{n} \geq \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$ *from* $(OF1)$. *Now, it follows from Proposition 1.2.9(#1) that* $n > 0$, *for any* $n \in \mathbb{N}$. *From Proposition 1.2.9(#7) and* $(OF1)$, $n > 0$ *and* $2 > 0$ *implies that* $\frac{2}{n} > 0$ *and* $\frac{2}{n} - \frac{1}{4} \geq -\frac{1}{4}$. *Similarly, from Proposition 1.2.9(#3) and* $(OF1)$, $\frac{2}{n} > 0$ *implies that* $-\frac{2}{n} < 0$ *and* $\frac{1}{4} - \frac{2}{n} < \frac{1}{4} + 0 = \frac{1}{4}$.

Combining our observations, we have that

$$(\forall n)\left[\left(n \in \mathbb{N} - \{1,2,3\} \wedge 2 \nmid n\right) \Rightarrow -\frac{1}{4} \leq x_n \leq \frac{1}{4}\right]$$

and

$$(\forall n)\left[\left(n \in \mathbb{N} - \{1,2,3\} \wedge 2 \mid n\right) \Rightarrow -\frac{1}{4} \leq x_n \leq \frac{1}{4}\right].$$

Finally, $x_1 = \frac{7}{4}$, $x_2 = -\frac{3}{4}$, *and* $x_3 = \frac{5}{12}$, *each of which is outside of* $\left[-\frac{1}{4}, \frac{1}{4}\right]$. *Comparing the values leads to the conclusion that* $\operatorname{glb}(A) = -\frac{3}{4}$.

Excursion 1.2.28 *Find* $\operatorname{glb}(A)$ *for each of the following. Since your goal is simply to find the greatest lower bound, you can use any pre-advanced calculus information that is helpful.*

1. $A = \left\{\frac{3 + (-1)^n}{2^{n+1}} : n \in \mathbb{J}\right\}$

2. $A = \left\{ \frac{1}{2^n} + \frac{1}{3^m} : n, m \in \mathbb{N} \right\}$

Our earlier discussion in Excursion 1.2.21, the set given in (1) leads to the conclusion that $\operatorname{glb}(A) = 0$. For (2), note that each of $\frac{1}{2^n}$ and $\frac{1}{3^m}$ are strictly decreasing to 0 as n and m are increasing, respectively. This leads us to conclude that $\operatorname{glb}(A) = 0$; although it was not requested, we note that $\sup(A) = \frac{5}{6}$.

We close this section with a theorem that relates least upper bounds and greatest lower bounds.

Theorem 1.2.29 *Suppose* $(S, <)$ *is an ordered set with the least upper bound property and that* B *is a nonempty subset of* S *that is bounded below. Let*

$$L = \{g \in S : (\forall a)\,(a \in B \Rightarrow g \leq a)\}.$$

Then $\alpha = \sup(L)$ *exists in* S*, and* $\alpha = \inf(B)$.

Proof. Suppose that $(S, <)$ is an ordered set with the least upper bound property and that B is a nonempty subset of S that is bounded below. Then

$$L = \{g \in S : (\forall a)\,(a \in B \Rightarrow g \leq a)\}.$$

is not empty. Note that for each $b \in B$ we have that $g \leq b$ for all $g \in L$; i.e., each element of B is an upper bound for L. Since $L \subset S$ is bounded above and S satisfies the least upper bound property, the least upper bound of L exists and is in S. Let $\alpha = \sup(L)$.

Now we want to show that α is the greatest lower bound for B.

■

Definition 1.2.30 *An ordered set* $(S, <)$ *has the* ***greatest lower bound property*** *if and only if*

$$(\forall E)\left[\begin{array}{c}(E \subset S \wedge E \neq \emptyset \wedge (\exists \gamma)(\gamma \in S \wedge (\forall a)(a \in E \Rightarrow \gamma \leq a))) \\ \Rightarrow ((\exists w)(w = \operatorname{glb}(E) \wedge w \in S))\end{array}\right].$$

Remark 1.2.31 *Theorem 1.2.29 tells us that every ordered set that satisfies the least upper bound property also satisfies the greatest lower bound property.*

1.3 The Real Field

The Appendix for Chapter 1 of our text offers a construction of "the reals" from "the rationals". In our earlier observation of special subsets of an ordered field, we offered formal definitions of the natural numbers of a field, the integers of a field, and the rationals of a field. Notice that the definitions were not tied to the objects (symbols) that we already accept as numbers. It is not the form of the objects in the ordered field that is important; it is the set of properties that must be satisfied. Once we accept the existence of an ordered field, all ordered fields are alike. While this identification of ordered fields and their corresponding special subsets can be made more formal, we will not seek that formalization.

It is interesting that our mathematics education actually builds up to the formulation of the real number field. Of course, the presentation is more hands-on and intuitive. At this point, we accept our knowledge of sums and products involving real numbers. I want to highlight parts of the building process simply to put the properties in perspective and to relate the least upper bound property to something tangible. None of this part of the discussion is rigorous. First, define the symbols 0 and 1, by $\{\} \underset{def}{=} 0$ and $\{\emptyset\} \underset{def}{=} 1$ and suppose that we have an ordered field $(R, +, \cdot, 0, 1, \leq)$. Furthermore, picture a representation of a straight horizontal line $(\longleftarrow\!\!\!-\!\!\!\longrightarrow)$ on which we will place elements of this field in a way that attaches some geometric meaning to their location. The natural numbers of this field $\mathbb{N}_R$ is the "smallest" inductive subset; it is closed under addition and multiplication. It can be proved (Some of you saw the proofs in your MAT108 course.) that

$$(\forall x)\,(x \in \mathbb{N}_R \Rightarrow x \geq 1)$$

and

$$(\forall w)\,(w \in \mathbb{N}_R \Rightarrow \neg\,(\exists v)\,(v \in \mathbb{N}_R \wedge w < v < w + 1))\,.$$

This motivates our first set of markings on the representative line. Let's indicate the first mark as a "place for 1." Then the next natural number of the field is $1+1$, while the one after that is $(1 + 1) + 1$, followed by $[(1 + 1) + 1] + 1$, etc. This naturally leads us to choose a fixed length to represent 1 (or "1 unit") and place a mark for each successive natural number 1 away from and to the right of the previous natural number. It doesn't take too long to see that our collections of "added $1's$" is not a pretty or easy to read labelling system; this motivates our desire for neater representations. The symbols that we have come to accept are 1, 2, 3, 4, 5, 6, 7, 8, and 9. In the space provided draw a picture that indicates what we have thus far.

The fact that, in an ordered field, $0 < 1$ tells us to place 0 to the left of 1 on our representative line; then $0 + 1 = \{\ \} \cup \{\emptyset\} = \{\emptyset\} = 1$ justifies placing 0 "1 unit" away from the 1. Now the definition of the integers of a field $\mathbb{Z}_R$ adjoins the additive inverses of the natural numbers of a field; our current list of natural numbers leads to acceptance of $-1, -2, -3, -4, -5, -6, -7, -8$, and -9 as labels of the markings of the new special elements and their relationship to the natural numbers mandates their relative locations. Use the space provided to draw a picture that indicates what we have thus far.

Your picture should show several points with each neighboring pair having the same distance between them and "lots of space" with no labels or markings, but we still have the third special subset of the ordered field; namely, the rationals of the field $\mathbb{Q}_R$. We are about to prove an important result concerning the "density of the rationals" in an ordered field. But, for this intuitive discussion, our "grade school knowledge" of fractions will suffice. Picture (or use the last picture that you drew to illustrate) the following process: Mark the midpoint of the line segment from 0 to 1 and label it 2^{-1} or $\frac{1}{2}$; then mark the midpoint of each of the smaller line segments (the one from 0 to $\frac{1}{2}$ and the one from $\frac{1}{2}$ to 1) and label the two new points $\frac{1}{4}$ and $\frac{3}{4}$, respectively; repeat the process with the four smaller line segments to get $\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}$ as the marked rationals between 0 and 1. It doesn't take too many iterations of this process to have it look like you have filled the interval. Of course, we know that we haven't because any rational in the from $p \cdot q^{-1}$ where $0 < p < q$ and $q \neq 2^n$ for any n has been omitted. What turned out to be a surprise, at the time of discovery, is that all the rationals r such the $0 \leq r \leq 1$ will not be "enough to fill the interval $[0, 1]$." At this point we have the set of elements of the field that are not in any of the special subsets, $R - \mathbb{Q}_R$, and the "set of vacancies" on our model line. We don't know that there is a one-to-one correspondence between them. That there is a correspondence follows from the what is proved in the Appendix to Chapter 1 of our text.

Henceforth, we use $(\mathbb{R}, +, \cdot, 0, 1, <)$ to denote the ordered field (of reals) that satisfies the least upper bound property and may make free use of the fact that for any $x \in \mathbb{R}$ we have that x is either rational or the least upper bound of a set of rationals. Note that the subfield $(\mathbb{Q}, +, \cdot, 0, 1, <)$ is an ordered field that does not satisfy the least upper bound property.

1.3.1 Density Properties of the Reals

In this section we prove some useful density properties for the reals.

Lemma 1.3.1 *If $S \subseteq \mathbb{R}$ has L as a least upper bound L, then*

$$(\forall \varepsilon)\left((\varepsilon \in \mathbb{R} \wedge \varepsilon > 0) \Rightarrow (\exists s)\left(s \in S \wedge L - \varepsilon < s \leq L\right)\right).$$

Proof. Suppose S is a nonempty subset of $\mathbb{R}$ such that $L = \sup(S)$ and let $\varepsilon \in \mathbb{R}$ be such that $\varepsilon > 0$. By Proposition 1.2.9(#3) and $(OF1)$, $-\varepsilon < 0$ and $L - \varepsilon < L$. From the definition of least upper bound, each upper bound of S is greater than or equal to L. Hence, $L - \varepsilon$ is not an upper bound for S from which we conclude that $\neg(\forall s)(s \in S \Rightarrow s \leq L - \varepsilon)$ is satisfied; i.e.,

$$(\exists s)(s \in S \wedge L - \varepsilon < s).$$

Combining this with $L = \sup(S)$ yields that

$$(\exists s)(s \in S \wedge L - \varepsilon < s \leq L).$$

Since ε was arbitrary, $(\forall \varepsilon)((\varepsilon \in \mathbb{R} \wedge \varepsilon > 0) \Rightarrow (\exists s)(s \in S \wedge L - \varepsilon < s \leq L))$ as claimed. ■

Theorem 1.3.2 (The Archimedean Principle for Real Numbers) *If α and β are positive real numbers, then there is some positive integer n such that $n\alpha > \beta$.*

Proof. The proof will be by contradiction. Suppose that there exist positive real numbers α and β such that $n\alpha \leq \beta$ for every natural number n. Since $\alpha > 0$, $\alpha < 2\alpha < 3\alpha < \cdots < n\alpha < \cdots$ is an increasing sequence of real numbers that is bounded above by β. Since $(\mathbb{R}, \leq)$ satisfies the least upper bound property $\{n\alpha : n \in \mathbb{N}\}$ has a least upper bound in $\mathbb{R}$, say L. Choose $\epsilon = \frac{1}{2}\alpha$ which is positive because $\alpha > 0$. Since $L = \sup\{n\alpha : n \in \mathbb{N}\}$, from Lemma 1.3.1, there exists $s \in \{n\alpha : n \in \mathbb{N}\}$ such that $L - \varepsilon < s \leq L$. If $s = N\alpha$, then for all natural numbers $m > N$, we also have that $L - \varepsilon < m\alpha \leq L$. Hence, for $m > N$, $0 \leq L - m\alpha < \varepsilon$. In particular,

$$0 \leq L - (N+1)\alpha < \varepsilon = \frac{1}{2}\alpha$$

and

$$0 \leq L - (N+2)\alpha < \varepsilon = \frac{1}{2}\alpha.$$

Thus, $L - \frac{1}{2}\alpha < (N+1)\alpha$ and $(N+2)\alpha < L < L + \frac{1}{2}\alpha$. But adding α to both sides of the first inequality, yields $L + \frac{1}{2}\alpha < (N+2)\alpha$ which contradicts $(N+2)\alpha < L + \frac{1}{2}\alpha$. Hence, contrary to our original assumption, there exists a natural number n such that $n\alpha > \beta$. ■

Corollary 1.3.3 (Density of the Rational Numbers) *If α and β are real numbers with $\alpha < \beta$, then there is a rational number r such that $\alpha < r < \beta$.*

Proof. Since 1 and $\beta - \alpha$ are positive real numbers, by the Archimedean Principle, there exists a positive integer m such that $1 < m(\beta - \alpha)$, or equivalently

$$m\alpha + 1 \leq m\beta.$$

Let n be the largest integer such that $n \leq m\alpha$. It follows that

$$n + 1 \leq m\alpha + 1 \leq m\beta.$$

Since n is the largest integer such that $n \leq m\alpha$, we know that $m\alpha < n + 1$. Consequently, $m\alpha < n + 1 < m\beta$, which is equivalent to having

$$\alpha < \frac{n+1}{m} < \beta.$$

Therefore, we have constructed a rational number that is between α and β. ■

Corollary 1.3.4 (Density of the Irrational Numbers) *If α and β are real numbers with $\alpha < \beta$, then there is an irrational number γ such that $\alpha < \gamma < \beta$.*

Proof. Suppose that α and β are real numbers with $\alpha < \beta$. By Corollary 1.3.3, there is a rational r that is between $\frac{\alpha}{\sqrt{2}}$ and $\frac{\beta}{\sqrt{2}}$. Since $\sqrt{2}$ is irrational, we conclude that $\gamma = r \cdot \sqrt{2}$ is an irrational that is between α and β. ■

1.3.2 Existence of nth Roots

The primary result in this connection that is offered by the author of our text is the following

Theorem 1.3.5 *For $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, we have that*

$$(\forall x)\,(\forall n)\left(x \in \mathbb{R}^+ \wedge n \in \mathbb{J} \Rightarrow (\exists! y)\left(y \in \mathbb{R} \wedge y^n = x\right)\right).$$

Before we start the proof, we note the following fact that will be used in the presentation.

Fact 1.3.6 $(\forall y)\,(\forall z)\,(\forall n)\left[(y,z \in \mathbb{R} \wedge n \in \mathbb{J} \wedge 0 < y < z) \Rightarrow y^n < z^n\right]$

To see this, for $y, z \in \mathbb{R}$ *satisfying are* $0 < y < z$, *let*

$$S = \left\{n \in \mathbb{J} : y^n < z^n\right\}.$$

Our set-up automatically places $1 \in S$. *Suppose that* $k \in S$*; i.e.,* $k \in \mathbb{J}$ *and* $y^k < z^k$. *Since* $0 < y$, *by* $(OF2)$, $y^{k+1} = y \cdot y^k < y \cdot z^k$. *From* $0 < z$ *and repeated use of Proposition 1.2.9(#2), we can justify that* $0 < z^k$. *Then* $(OF2)$ *with* $0 < z^k$ *and* $y < z$ *yields that* $y \cdot z^k < z \cdot z^k = z^{k+1}$. *As a consequence of the transitive law,*

$$y^{k+1} < y \cdot z^k \wedge y \cdot z^k < z^{k+1} \Rightarrow y^{k+1} < z^{k+1};$$

that is, $k+1 \in S$. *Since* k *was arbitrary, we conclude that* $(\forall k)\,(k \in S \Rightarrow (k+1) \in S)$.

From $1 \in S \wedge (\forall k)\,(k \in S \Rightarrow (k+1) \in S)$, S *is an inductive subset of the natural numbers. By the Principle of Mathematical Induction (PMI),* $S = \mathbb{J}$. *Since* y *and* z *were arbitrary, this completes the justification of the claim.*

Fact 1.3.7 $(\forall w)\,(\forall n)\left[(w \in \mathbb{R} \wedge n \in \mathbb{J} - \{1\} \wedge 0 < w < 1) \Rightarrow w^n < w\right]$

Since $n \geq 2$, $n - 1 \geq 1$ *and, by Fact 1.3.6,* $w^{n-1} < 1^{n-1} = 1$. *From* $(OF2)$, $0 < w \wedge w^{n-1} < 1$ *implies that* $w^n = w^{n-1} \cdot w < 1 \cdot w = w$*; i.e.,* $w^n < w$ *as claimed.*

Fact 1.3.8 $(\forall a)\,(\forall b)\,(\forall n)\,[(a,b \in \mathbb{R} \wedge n \in \mathbb{J} - \{1\} \wedge 0 < a < b)$
$\Rightarrow (b^n - a^n) < (b-a)\,nb^{n-1}]$

From Fact 1.3.6, $n \geq 2 \wedge 0 < a < b \Rightarrow a^{n-1} < b^{n-1}$, *while* $(OF2)$ *yields that* $a \cdot b^j < b \cdot b^j = b^{j+1}$ *for* $j = 1, 2, ..., n-2$. *It can be shown (by repeated application of Exercise 6(a)) that*

$$b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1} < b^{n-1} + b^{n-1} + \cdots + b^{n-1} = nb^{n-1};$$

this, with $(OF2)$, *implies that*

$$b^n - a^n = (b-a)\left(b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1}\right) < (b-a)\,nb^{n-1}$$

as claimed.

Proof. (of the theorem.) Let $\mathbb{R}^+ = \{u \in \mathbb{R} : u > 0\}$. When $n = 1$, there is nothing to prove so we assume that $n \geq 2$. For fixed $x \in \mathbb{R}^+$ and $n \in \mathbb{J} - \{1\}$, set

$$E = \left\{t \in \mathbb{R}^+ : t^n < x\right\}.$$

Excursion 1.3.9 *Use* $w = \dfrac{x}{1+x}$ *to justify that* $E \neq \emptyset$.

Now let $u = 1 + x$ and suppose that $t > u > 0$. Fact 1.3.6 yields that $t^n > u^n$. From Proposition 1.2.9(#7), $u > 1 \Rightarrow 0 < \dfrac{1}{u} < 1$. It follows from Fact 1.3.7 and Proposition 1.2.9(#7) that $0 < \dfrac{1}{u^n} \leq \dfrac{1}{u}$ and $u^n \geq u$. By transitivity, $t^n > u^n \wedge u^n \geq u$ implies that $t^n > u$. Finally, since $u > x$ transitivity leads to the conclusion that $t^n > x$. Hence, $t \notin E$. Since t was arbitrary, $(\forall t)\,(t > u \Rightarrow t \notin E)$ which is equivalent to $(\forall t)\,(t \in E \Rightarrow t \leq u)$. Therefore, $E \subset \mathbb{R}$ is bounded above. From the least upper bound property, $\operatorname{lub}(E)$ exists. Let

$$y = \operatorname{lub}(E).$$

Since $E \subset \mathbb{R}^+$, we have that $y \geq 0$.

By the Trichotomy Law, one and only one of $y^n = x$, $y^n < x$, or $y^n > x$. In what follows we will that neither of the possibilities $y^n < x$, or $y^n > x$ can hold.

Case 1: If $y^n < x$, then $x - y^n > 0$. Since $y + 1 > 0$ and $n \geq 1$, $\dfrac{x - y^n}{n\,(y+1)^{n-1}} > 0$ and we can choose h such that $0 < h < 1$ and

$$h < \frac{x - y^n}{n\,(y+1)^{n-1}}.$$

Taking $a = y$ and $b = y + h$ in Fact 1.3.8 yields that

$$(y+h)^n - y^n < hn\,(y+h)^{n-1} < x - y^n.$$

Excursion 1.3.10 *Use this to obtain contradict that* $y = \sup(E)$.

Case 2: If $0 < x < y^n$, then $0 < y^n - x < ny^n$. Hence,

$$k = \frac{y^n - x}{ny^{n-1}}$$

is such that $0 < k < y$. For $t \geq (y - k)$, Fact 1.3.6 yields that $t^n \geq (y - k)^n$.

From Fact 1.3.8, with $b = y$ and $a = y - k$, we have that

$$y^n - t^n \leq y^n - (y - k)^n < kny^{n-1} = y^n - x.$$

Excursion 1.3.11 *Use this to obtain another contradiction.*

From case 1 and case 2, we conclude that $y^n = x$. this concludes the proof that there exists a solution to the given equation.

The uniqueness of the solution follows from Fact 1.3.6. To see this, note that, if $y^n = x$ and w is such that $0 < w \neq y$, then $w < y$ implies that $w^n < y^n = x$, while $y < w$ implies that $x = y^n < w^n$. In either case, $w^n \neq x$. ■

*******For Excursion 1.3.9, you want to justify that the given w is in E. Because $0 < x < 1+x$, $0 < w = \dfrac{x}{1+x} < 1$. In view of fact 1.3.7, $w^n < w$ for $n \geq 2$ or $w^n \leq w$ for $n \geq 1$. But $x > 0 \wedge 1 + x > 1$ implies that $\dfrac{1}{1+x} < 1 \wedge \dfrac{x}{1+x} < x \cdot 1 = x$. From transitivity, $w^n < w \wedge w < x \Rightarrow w^n < x$; i.e., $w \in E$.

To obtain the desired contradiction for completion of Excursion 1.3.10, hopefully you notices that the given inequality implied that $(y + h)^n < x$ which would place $y+h$ in E; since $y+h > y$, this would contradict that $y = \sup(E)$ from which we conclude that $y^n < x$ is not true.

The work needed to complete Excursion 1.3.11 was a little more involved. In this case, the given inequality led to $-t^n < -x$ or $t^n > x$ which justifies that $t \notin E$; hence, $t > y-k$ implies that $t \notin E$ which is logically equivalent to $t \in E$ implies that $t < y-k$. This would make $y-k$ an upper bound for E which is a contradiction. Obtaining the contradiction yields that $x < y^n$ is also not true.***

Remark 1.3.12 *For x a positive real number and n a natural number, the number y that satisfies the equation $y^n = x$ is written as $\sqrt[n]{x}$ and is read as "the nth root of x."*

Repeated application of the associativity and commutativity of multiplication can be used to justify that, for positive real numbers α and β and n a natural number,

$$\alpha^n \beta^n = (\alpha\beta)^n .$$

From this identity and the theorem we have the following identity involving nth roots of positive real numbers.

Corollary 1.3.13 *If a and b are positive real numbers and n is a positive integer, then*

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$

Proof. For $\alpha = a^{1/n}$ and $\beta = b^{1/n}$, we have that $ab = \alpha^n \beta^n = (\alpha\beta)^n$. Hence $\alpha\beta$ is the unique solution to $y^n = ab$ from which we conclude that $(ab)^{1/n} = \alpha\beta$ as needed. ■

1.3.3 The Extended Real Number System

The extended real number system is $\mathbb{R} \cup \{-\infty, +\infty\}$ where $(\mathbb{R}, +, \cdot, 0, 1, <)$ is the ordered field that satisfies the least upper bound property as discussed above and the symbols $-\infty$ and $+\infty$ are defined to satisfy $-\infty < x < +\infty$ for all $x \in \mathbb{R}$. With this convention, any nonempty subset S of the extended real number system is bounded above by $+\infty$ and below by $-\infty$; if S has no finite upper bound, we write $\text{lub}(S) = +\infty$ and when S has no finite lower limit, we write $\text{glb}(S) = -\infty$.

The $+\infty$ and $-\infty$ are useful symbols; they are not numbers. In spite of their appearance, $-\infty$ is not an additive inverse for $+\infty$. This means that there is no meaning attached to any of the expressions $\infty - \infty$ or $\frac{\infty}{-\infty}$ or $\frac{\infty}{\infty}$; in fact, these expressions should never appear in things that you write. Because the symbols ∞ and $-\infty$ do not have additive (or multiplicative) inverses, $\mathbb{R} \cup \{-\infty, \infty\}$ is not a field. On the other hand, we do have some conventions concerning “interaction” of the special symbols with elements of the field $\mathbb{R}$; namely,

- If $x \in \mathbb{R}$, then $x + \infty = +\infty$, $x - \infty = -\infty$ and $\frac{x}{\infty} = \frac{x}{-\infty} = 0$.
- If $x > 0$, then $x \cdot (+\infty) = +\infty$ and $x \cdot (-\infty) = -\infty$.
- If $x < 0$, then $x \cdot (+\infty) = -\infty$ and $x \cdot (-\infty) = +\infty$.

Notice that nothing is said about the product of zero with either of the special symbols.

1.4 The Complex Field

For $\mathbb{C} = \mathbb{R} \times \mathbb{R}$, define addition $(+)$ and multiplication $(\cdot)$ by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)\,,$$

respectively. That addition and multiplication are binary operations on $\mathbb{C}$ is a consequence of the closure of $\mathbb{R}$ under addition and multiplication. It follows immediately that

$$(x, y) + (0, 0) = (x, y) \qquad \textit{and} \qquad (x, y) \cdot (1, 0) = (x, y)\,.$$

Hence, $(0, 0)$ and $(1, 0)$ satisfy the additive identity property and the multiplicative identity field property, respectively. Since the binary operations are defined as combinations of sums and products involving reals, direct substitution and appropriate manipulation leads to the conclusion that addition and multiplication over $\mathbb{C}$ are commutative and associative under addition and multiplication. (The actual manipulations are shown in our text on pages 12-13.)

To see that the additive inverse property is satisfied, note that $(x, y) \in \mathbb{C}$ implies that $x \in \mathbb{R} \wedge y \in \mathbb{R}$. The additive inverse property in the field $\mathbb{R}$ yields that $-x \in \mathbb{R}$

and $-y \in \mathbb{R}$. It follows that $(-x,-y) \in \mathbb{C}$ and $(x,y)+(-x,-y)=(0,0)$ and needed.

Suppose $(x,y) \in \mathbb{C}$ is such that $(x,y) \neq (0,0)$. Then $x \neq 0 \vee y \neq 0$ from which we conclude that $x^2+y^2 \neq 0$ and $(a,b) \underset{def}{=} \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ is well defined. Now,

$$\begin{aligned}
(x,y)\cdot(a,b) &= (x,y)\cdot\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)\\
&= \left(x\cdot\frac{x}{x^2+y^2} - y\cdot\frac{-y}{x^2+y^2}, x\cdot\frac{-y}{x^2+y^2} + y\cdot\frac{x}{x^2+y^2}\right)\\
&= \left(\frac{x\cdot x+(-y)\cdot(-y)}{x^2+y^2}, \frac{x\cdot(-y)+y\cdot x}{x^2+y^2}\right)\\
&= \left(\frac{x^2+y^2}{x^2+y^2}, \frac{-xy+yx}{x^2+y^2}\right)\\
&= (1,0).
\end{aligned}$$

Hence, the multiplicative inverse property is satisfied for $(\mathbb{C},+,\cdot)$.

Checking that the distributive law is satisfied is a matter of manipulating the appropriate combinations over the reals. This is shown in our text on page 13.

Combining our observations justifies that $(\mathbb{C},+,\cdot,(0,0),(1,0))$ is a field. It is known as the complex field or the field of complex numbers.

Remark 1.4.1 *Identifying each element of* $\mathbb{C}$ *in the form* $(x,0)$ *with* $x \in \mathbb{R}$ *leads to the corresponding identification of the sums and products,* $x+a=(x,0)+(a,0)=(x+a,0)$ *and* $x\cdot a=(x,0)\cdot(a,0)=(x\cdot a,0)$. *Hence, the real field is a subfield of the complex field.*

The following definition will get us to an alternative formulation for the complex numbers that can make some of their properties easier to remember.

Definition 1.4.2 *The complex number* $(0,1)$ *is defined to be* i.

With this definition, it can be shown directly that

- $i^2=(-1,0)=-1$ and
- if a and b are real numbers, then $(a,b)=a+bi$.

With these observations we can write

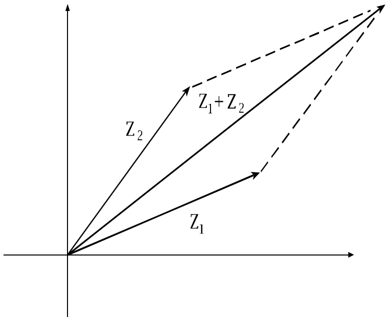
$$\mathbb{C} = \left\{ a + bi : a, b \in \mathbb{R} \wedge i^2 = -1 \right\}$$

with addition and multiplication being carried out using the distributive law, commutativity, and associativity.

We have two useful forms for complex numbers; the rectangular and trigonometric forms for the complex numbers are freely interchangeable and offer different geometric advantages.

From Rectangular Coordinates

Complex numbers can be represented geometrically as points in the plane. We plot them on a rectangular coordinate system that is called an Argand Graph. In $z = x + iy$, x is the real part of z, denoted by $\operatorname{Re} z$, and y is the imaginary part of z, denoted by $\operatorname{Im} z$. When we think of the complex number $x + iy$ as a vector $\overrightarrow{OP}$ joining the origin $O = (0, 0)$ to the point $P = (x, y)$, we grasp the natural geometric interpretation of addition $(+)$ in $\mathbb{C}$.



Definition 1.4.3 *The **modulus** of a complex number z is the magnitude of the vector representation and is denoted by* $|z|$. *If* $z = x + iy$, *then* $|z| = \sqrt{x^2 + y^2}$.

Definition 1.4.4 *The **argument** of a nonzero complex number* z, *denoted by* $\arg z$, *is a measurement of the angle that the vector representation makes with the positive real axis.*

Definition 1.4.5 *For* $z = x + iy$, *the **conjugate** of* z, *denoted by* $\bar{z}$, *is* $x - iy$.

Most of the properties that are listed in the following theorems can be shown fairly directly from the rectangular form.

Theorem 1.4.6 *For z and w complex numbers,*

1. $|z| \geq 0$ *with equality only if* $z = 0$,
2. $|\overline{z}| = |z|$,
3. $|zw| = |z|\,|w|$,
4. $|\operatorname{Re} z| \leq |z|$ *and* $|\operatorname{Im} z| \leq |z|$,
5. $|z + w|^2 = |z|^2 + 2 \operatorname{Re} z\overline{w} + |w|^2$.

The proofs are left as exercises.

Theorem 1.4.7 (The Triangular Inequalities) *For complex numbers z_1 and z_2,*

$$|z_1 + z_2| \leq |z_1| + |z_2|\,, \text{ and } |z_1 - z_2| \geq ||z_1| - |z_2||\,.$$

Proof. To see the first one, note that

$$\begin{aligned} |z_1 + z_2|^2 \quad &= \quad |z_1|^2 + 2 \operatorname{Re} z_1 z_2 + |z_2|^2 \\ &\leq \quad |z_1|^2 + 2\,|z_1|\,|z_2| + |z_2|^2 \quad = \quad (|z_1| + |z_2|)^2 \end{aligned} .$$

The proof of the second triangular inequality is left as an exercise. ■

Theorem 1.4.8 *If z and w are complex numbers, then*

1. $\overline{z + w} = \bar{z} + \bar{w}$
2. $\overline{zw} = \bar{z}\bar{w}$
3. $\operatorname{Re} z = \dfrac{z + \bar{z}}{2}$, $\operatorname{Im} z = \dfrac{z - \bar{z}}{2i}$,
4. $z\bar{z}$ *is a nonnegative real number.*

From Polar Coordinates

For nonzero $z = x + iy \in \mathbb{C}$, let $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\dfrac{y}{x}\right) = \arg z$. Then the trigonometric form is

$$z = r\,(\cos\theta + i\,\sin\theta)\,.$$

In engineering, it is customary to use $\operatorname{cis}\theta$ for $\cos\theta + i\,\sin\theta$ in which case we write $z = r\,\operatorname{cis}\theta$.

NOTE: While (r, θ) uniquely determines a complex number, the converse is not true.

Excursion 1.4.9 *Use the polar form for complex numbers to develop a geometric interpretation of the product of two complex numbers.*

The following identity can be useful when working with complex numbers in polar form.

Proposition 1.4.10 (DeMoivre's Law) *For θ real and $n \in \mathbb{Z}$,*

$$[\operatorname{cis}\theta]^n = \operatorname{cis} n\theta.$$

Example 1.4.11 *Find all the complex numbers that when cubed give the value one.*

We are looking for all $\zeta \in \mathbb{C}$ such that $\zeta^3 = 1$. DeMoivre's Law offers us a nice tool for solving this equation. Let $\zeta = r \operatorname{cis}\theta$. Then$\zeta^3 = 1 \Leftrightarrow r^3 \operatorname{cis} 3\theta = 1$. Since $\left|r^3 \operatorname{cis} 3\theta\right| = r^3$, we immediately conclude that we must have $r = 1$. Hence, we need only solve the equation $\operatorname{cis} 3\theta = 1$. Due to the periodicity of the sine and cosine, we know that the last equation is equivalent to finding all θ such that $\operatorname{cis} 3\theta = \operatorname{cis}(2k\pi)$ for $k \in \mathbb{Z}$ which yields that $3\theta = 2k\pi$ for $k \in \mathbb{Z}$. But $\left\{\frac{2k\pi}{3} : k \in \mathbb{Z}\right\} = \left\{-\frac{2\pi}{3}, 0, \frac{2\pi}{3}\right\}$. Thus, we have three distinct complex numbers whose cubes are one; namely, $\operatorname{cis}\left(-\frac{2\pi}{3}\right)$, $\operatorname{cis} 0 = 1$, and $\operatorname{cis}\left(\frac{2\pi}{3}\right)$. In rectangular form, the three complex numbers whose cubes are one are: $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$, 0, and $-\frac{1}{2} + \frac{\sqrt{3}}{2}$.

Theorem 1.4.12 (Schwarz's Inequality) *If $a_1, ..., a_n$ and $b_1, ..., b_n$ are complex numbers, then*

$$\left|\sum_{j=1}^{n} a_j \overline{b}_j\right|^2 \leq \left(\sum_{j=1}^{n} \left|a_j\right|^2\right)\left(\sum_{j=1}^{n} \left|b_j\right|^2\right).$$

Proof. First the statement is certainly true if $b_k = 0$ for all k, $1 \le k \le n$. Thus we assume that not all the b_k are zero. Now, for any $\lambda \in \mathbb{C}$, note that

$$\sum_{j=1}^{n} \left| a_j - \lambda \overline{b_j} \right|^2 \ge 0.$$

Excursion 1.4.13 *Make use of this inequality and the choice of*

$$\lambda = \left(\sum_{j=1}^{n} a_j b_j \right) \left(\sum_{j=1}^{n} \left| b_j \right|^2 \right)^{-1}$$

to complete the proof.

■

Remark 1.4.14 *A special case of Schwarz's Lemma contains information relating the modulus of two vectors with the absolute value of their dot product. For example, if* $\overrightarrow{v_1} = (a_1, a_2)$ *and* $\overrightarrow{v_2} = (b_1, b_2)$ *are vectors in* $\mathbb{R} \times \mathbb{R}$, *then Schwarz's Lemma merely reasserts that* $\left| \overrightarrow{v_1} \bullet \overrightarrow{v_2} \right| = |a_1 b_1 + a_2 b_2| \le \left| \overrightarrow{v_1} \right| \left| \overrightarrow{v_2} \right|$.

1.4.1 Thinking Complex

Complex variables provide a very convenient way of describing shapes and curves. It is important to gain a facility at representing sets in terms of expressions involving complex numbers because we will use them for mappings and for applications to various phenomena happening within "shapes." Towards this end, let's do some work on describing sets of complex numbers given by equations involving complex variables.

One way to obtain a description is to translate the expressions to equations involving two real variables by substituting $z = x + iy$.

Example 1.4.15 *Find all complex numbers z that satisfy*

$$2\,|z| = 2\operatorname{Im} z - 1.$$

Let $z = x + iy$. *Then*

$$\begin{aligned}
2\,|z| = 2\operatorname{Im} z - 1 \quad &\Leftrightarrow \quad 2\sqrt{x^2 + y^2} = 2y - 1 \\
&\Leftrightarrow \quad \left(4\left(x^2 + y^2\right) = 4y^2 - 4y + 1\right) \wedge \left(y \geq \frac{1}{2}\right) \\
&\Leftrightarrow \quad 4x^2 = -4y + 1 \wedge y \geq \frac{1}{2} \\
&\Leftrightarrow \quad x^2 = -\left(y - \frac{1}{4}\right) \wedge y \geq \frac{1}{2}.
\end{aligned}$$

The last equation implies that $y \leq \frac{1}{4}$. *Since* $y \leq \frac{1}{4} \wedge y \geq \frac{1}{2}$ *is never satisfied, we conclude that the set of solutions for the given equation is empty.*

Excursion 1.4.16 *Find all* $z \in \mathbb{C}$ *such that* $|z| - z = 1 + 2i$.

Your work should have given the $\frac{3}{2} - 2i$ as the only solution.

Another way, which can be quite a time saver, is to reason by TRANSLATING TO THE GEOMETRIC DESCRIPTION. In order to do this, there are some geometric descriptions that are useful for us to recall:

$\{z : \lvert z - z_0\rvert = r\}$	is the locus of all points z equidistant from the fixed point, z_0, with the distance being $r > 0$. (a circle)
$\{z : \lvert z - z_1\rvert = \lvert z - z_2\rvert\}$	is the locus of all points z equidistant from two fixed points, z_1 and z_2. (the perpendicular bisector of the line segment joining z_1 and z_2.)
$\{z : \lvert z - z_1\rvert + \lvert z - z_2\rvert = \rho\}$ for a constant $\rho > \lvert z_1 - z_2\rvert$	is the locus of all points for which the sum of the distances from 2 fixed points, z_1 and z_2, is a constant greater than $\lvert z_1 - z_2\rvert$. (an ellipse)

Excursion 1.4.17 *For each of the following, without substituting $x+iy$ for z, sketch the set of points z that satisfy the given equations. Provide labels, names, and/or important points for each object.*

1. $\left|\dfrac{z - 2i}{z + 3 + 2i}\right| = 1$

2. $|z - 4i| + |z + 7i| = 12$

3. $|4z + 3 - i| \leq 3$

*******The equations described a straight line, an ellipse, and a disk, respectively. In set notation, you should have obtained $\left\{ x + iy \in \mathbb{C} : y = -\frac{3}{4}x - \frac{9}{8} \right\}$,
$\left\{ x + iy \in \mathbb{C} : \dfrac{x^2}{\left(\frac{23}{4}\right)} + \dfrac{\left(y + \frac{3}{2}\right)^2}{6^2} = 1 \right\}$, and
$\left\{ x + iy \in \mathbb{C} : \left(x + \frac{3}{4}\right)^2 + \left(y - \frac{1}{4}\right)^2 \leq \left(\frac{3}{4}\right)^2 \right\}$.***

Remark 1.4.18 *In general, if* k *is a positive real number and* $a, b \in \mathbb{C}$, *then*

$$\left\{ z \in \mathbb{C} : \left| \frac{z - a}{z - b} \right| = k, k \neq 1 \right\}$$

describes a circle.

Excursion 1.4.19 *Use the space below to justify this remark.*

*******Simplifying $\left|\frac{z-a}{z-b}\right| = k$ leads to

$$\left(1-k^2\right)|z|^2 - 2\operatorname{Re}\left(a\bar{z}\right) + 2k^2\operatorname{Re}\left(b\bar{z}\right) + \left(|a|^2 - k^2|b|^2\right)$$

from which the remark follows.*******

1.5 Problem Set A

1. For $\mathbb{F} = \{p, q, r\}$, let the binary operations of addition, $\oplus$, and multiplication, $\otimes$, be defined by the following tables.

$\oplus$	r	q	p
r	r	q	p
q	q	p	r
p	p	r	q

$\otimes$	r	q	p
r	r	r	r
q	r	q	p
p	r	p	q

   (a) Is there an additive identity for the algebraic structure $(\mathbb{F}, \oplus, \otimes)$? Briefly justify your position.

   (b) Is the multiplicative inverse property satisfied? If yes, specify a multiplicative inverse for each element of $\mathbb{F}$ that has one.

  (c) Assuming the notation from our field properties, find $(r \oplus q) \otimes \left(p \oplus p^{-1}\right)$.
  (d) Is $\{(p, p), (p, r), (q, q), (p, q), (r, r)\}$ a field ordering on $\mathbb{F}$? Briefly justify your claim.

2. For a field $(\mathbb{F}, +, \cdot, e, f)$, prove each of the following parts of Proposition 1.1.6.

   (a) The multiplicative identity of a field is unique.
   (b) The multiplicative inverse of any element in $\mathbb{F} - \{e\}$ is unique.

3. For a field $(\mathbb{F}, +, \cdot, e, f)$, prove each of the following parts of Proposition 1.1.8.

   (a) $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow -(a+b) = (-a) + (-b))$
   (b) $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow a \cdot (-b) = -(a \cdot b))$
   (c) $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow (-a) \cdot b = -(a \cdot b))$
   (d) $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow (-a) \cdot (-b) = a \cdot b)$
   (e) $(\forall a)(\forall b)\left(a, b \in \mathbb{F}- \{e\} \Rightarrow (a \cdot b)^{-1} = \left(a^{-1}\right)\left(b^{-1}\right)\right)$

4. For a field $(\mathbb{F}, +, \cdot, 0, 1)$, prove Proposition 1.1.10(#1):
   $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow (\exists! x)(x \in \mathbb{F} \wedge a + x = b))$

5. For a field $(\mathbb{F}, +, \cdot, 0, 1)$, show that, for $a, b, c \in \mathbb{F}$,

$$a - (b + c) = (a - b) - c \qquad \textit{and} \qquad a - (b - c) = (a - b) + c.$$

   Give reasons for each step of your demonstration.

6. For an ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$, prove that

   (a) $(\forall a)(\forall b)(\forall c)(\forall d)[(a, b, c, d \in \mathbb{F} \wedge a < b \wedge c < d) \Rightarrow a + c < b + d]$
   (b) $(\forall a)(\forall b)(\forall c)(\forall d)[(a, b, c, d \in \mathbb{F} \wedge 0 < a < b \wedge 0 < c < d) \Rightarrow ac < bd]$

7. For an ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$, prove each of the following

(a) $(\forall a)\,(\forall b)\,(\forall c)\,\left[(a, b, c \in \mathbb{F} \wedge c \neq 0) \Rightarrow \left(a \cdot c^{-1}\right) + \left(b \cdot c^{-1}\right) = (a+b) \cdot c^{-1}\right]$

(b) $(\forall a)\,(\forall b)\,(\forall c)\,(\forall d)\,[(a, c \in \mathbb{F} \wedge b, d \in \mathbb{F} - \{0\}) \Rightarrow b \cdot d \neq 0 \wedge \left(a \cdot b^{-1}\right) + \left(c \cdot d^{-1}\right) = (a \cdot d + b \cdot c) \cdot (b \cdot d)^{-1}]$

8. Find the least upper bound and the greatest lower bound for each of the following.

   (a) $\left\{\dfrac{n + (-1)^n}{n} : n \in \mathbb{N}\right\}$

   (b) $\left\{(-1)^n \left(\pi + \dfrac{1}{n}\right) : n \in \mathbb{N}\right\}$

   (c) $\left\{\dfrac{1}{m} + \dfrac{1}{n} : m, n \in \mathbb{J}\right\}$

   (d) $\left\{\dfrac{1}{1+x^2} : x \in \mathbb{R}\right\}$

   (e) $\left\{\dfrac{1}{3^n} + \dfrac{1}{5^{n-1}} : n \in \mathbb{J}\right\}$

   (f) $\left\{x + \dfrac{1}{x} : x \in \mathbb{R} - \{0\}\right\}$

   (g) $\left\{x + \dfrac{1}{x} : \dfrac{1}{2} < x < 2\right\}$

9. Let $(X, \leq)$ be an ordered set and $A \subseteq X$. Prove that, if A has a least upper bound in X, it is unique.

10. Suppose that $S \subseteq \mathbb{R}$ is such that $\inf(S) = M$. Prove that

$$(\forall \varepsilon)\,((\varepsilon \in \mathbb{R} \wedge \varepsilon > 0) \Rightarrow (\exists g)\,(g \in S \wedge M \leq g < M + \varepsilon))\,.$$

11. For $f(x) = \dfrac{2}{x} + \dfrac{1}{x^2}$, find

    (a) $\sup f^{-1}((-\infty, 3))$

    (b) $\inf f^{-1}((3, \infty))$

12. Suppose that $P \subset Q \subset \mathbb{R}$ and $P \neq \emptyset$. If P and Q are bounded above, show that $\sup(P) \leq \sup(Q)$.

13. Let $A = \left\{x \in \mathbb{R} : (x+2)(x-3)^{-1} < -2\right\}$. Find the $\sup(A)$ and the $\inf(A)$.

14. Use the Principle of Mathematical Induction to prove that, for $a \geq 0$ and n a natural number, $(1+a)^n \geq 1+na$.

15. Find all the values of
    - (a) $(-2,3)(4,-1)$.
    - (b) $(1+2i)[3(2+i)-2(3+6i)]$.
    - (c) $(1+i)^3$.
    - (d) $(1+i)^4$.
    - (e) $(1+i)^n-(1-i)^n$.

16. Show that the following expressions are both equal to one.
    - (a) $\left[\dfrac{-1+i\sqrt{3}}{2}\right]^3$
    - (b) $\left[\dfrac{-1-i\sqrt{3}}{2}\right]^3$

17. For any integers k and n, show that $i^n = i^{n+4k}$. How many distinct values can be assumed by i^n?

18. Use the Principle of Mathematical Induction to prove DeMoivre's Law.

19. If $z_1 = 3-4i$ and $z_2 = -2+3i$, obtain graphically and analytically
    - (a) $2z_1+4z_2$.
    - (b) $3z_1-2\overline{z_2}$.
    - (c) $z_1-\overline{z_2}-4$.
    - (d) $|z_1+z_2|$.
    - (e) $|z_1-z_2|$.
    - (f) $|2\overline{z_1}+3\overline{z_2}-1|$.

20. Prove that there is no ordering on the complex field that will make it an ordered field.

21. Carefully justify the following parts of Theorem 1.4.6. For z and w complex numbers,
    - (a) $|z| \geq 0$ with equality only if $z=0$,
    - (b) $|\overline{z}| = |z|$,
    - (c) $|zw| = |z|\,|w|$,

(d) $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$,

(e) $|z+w|^2 = |z|^2 + 2\operatorname{Re} z\overline{w} + |w|^2$.

22. Prove the "other" triangular inequality: For complex numbers z_1 and z_2, $|z_1 - z_2| \geq ||z_1| - |z_2||$.

23. Carefully justify the following parts of Theorem 1.4.8. If z and w are complex numbers, then

    (a) $\overline{z+w} = \bar{z} + \bar{w}$

    (b) $\overline{zw} = \bar{z}\bar{w}$

    (c) $\operatorname{Re} z = \dfrac{z+\bar{z}}{2}$, $\operatorname{Im} z = \dfrac{z-\bar{z}}{2i}$,

    (d) $z\bar{z}$ is a nonnegative real number.

24. Find the set of all $z \in \mathbb{C}$ that satisfy:

    (a) $1 < |z| \leq 3$.

    (b) $\left|\dfrac{z-3}{z+2}\right| = 1$.

    (c) $\operatorname{Re} z^2 > 0$.

    (d) $|z-1| + |z+1| = 2$.

    (e) $\operatorname{Im} z^2 > 0$.

    (f) $\left|\dfrac{z+2}{z-1}\right| = 2$.

    (g) $|z-2| + |z+2| = 5$.

    (h) $|z| = 1 + \operatorname{Re}(z)$.

25. When does $az + b\bar{z} + c = 0$ represent a line?

26. Prove that the vector z_1 is parallel to the vector z_2 if and only if $\operatorname{Im}(z_1\bar{z}_2) = 0$.

Chapter 2

From Finite to Uncountable Sets

A considerable amount of the material offered in this chapter is a review of terminology and results that were covered in MAT108. Our brief visit allows us to go beyond some of what we saw and to build a deeper understanding of some of the material for which a revisit would be beneficial.

2.1 Some Review of Functions

We have just seen how the concept of function gives precise meaning for binary operations that form part of the needed structure for a field. The other "big" use of function that was seen in MAT108 was with defining "set size" or cardinality. For precise meaning of what constitutes set size, we need functions with two additional properties.

Definition 2.1.1 *Let A and B be nonempty sets and $f : A \longrightarrow B$. Then*

1. *f is* ***one-to-one****, written $f : A \xrightarrow{1-1} B$, if and only if*

$$(\forall x)\,(\forall y)\,(\forall z)\,((x,z) \in f \wedge (y,z) \in f \Rightarrow x = y)\,,$$

2. *f is* ***onto****, written $f : A \twoheadrightarrow B$, if and only if*

$$(\forall y)\,(y \in B \Rightarrow (\exists x)\,(x \in A \wedge (x,y) \in f))\,,$$

3. *f is a* ***one-to-one correspondence****, written $f : A \overset{1-1}{\twoheadrightarrow} B$, if and only if f is one-to-one and onto.*

Remark 2.1.2 *In terms of our other definitions,* $f : A \longrightarrow B$ *is onto if and only if*

$$\operatorname{rng}(f) \underset{def}{=} \{y \in B : (\exists x)(x \in A \wedge (x, y) \in f)\} = B$$

which is equivalent to $f[A] = B$.

In the next example, the first part is shown for completeness and to remind the reader about how that part of the argument that something is a function can be proved. As a matter of general practice, as long as we are looking at basic functions that result in simple algebraic combinations of variables, you can assume that was is given in that form in a function on either its implied domain or on a domain that is specified.

Example 2.1.3 *For* $f = \left\{\left(x, \dfrac{x}{1-|x|}\right) \in \mathbb{R} \times \mathbb{R} : -1 < x < 1\right\}$, *prove that*

$$f : (-1, 1) \overset{1-1}{\twoheadrightarrow} \mathbb{R}.$$

(a) *By definition,* $f \subseteq \mathbb{R} \times \mathbb{R}$*; i.e.,* f *is a relation from* $(-1, 1)$ *to* $\mathbb{R}$.

Now suppose that $x \in (-1, 1)$. *Then* $|x| < 1$ *from which it follows that* $1 - |x| \neq 0$. *Hence,* $(1 - |x|)^{-1} \in \mathbb{R} - \{0\}$ *and* $y \underset{def}{=} x \cdot (1 - |x|)^{-1} \in \mathbb{R}$ *because multiplication is a binary operation on* $\mathbb{R}$. *Since* x *was arbitrary, we have shown that*

$$(\forall x)(x \in (-1, 1) \Rightarrow (\exists y)(y \in \mathbb{R} \wedge (x, y) \in f))\text{; i.e.,}$$

$\operatorname{dom}(f) = (-1, 1)$.

Suppose that $(x, y) \in f \wedge (x, v) \in f$. *Then* $u = x \cdot (1 - |x|)^{-1} = v$ *because multiplication is single-valued on* $\mathbb{R} \times \mathbb{R}$. *Since* x, u, *and* v *were arbitrary,*

$$(\forall x)(\forall u)(\forall v)((x, u) \in f \wedge (x, v) \in f \Rightarrow u = v)\text{;}$$

i.e., f *is single-valued.*

Because f *is a single-valued relation from* $(-1, 1)$ *to* $\mathbb{R}$ *whose domain is* $(-1, 1)$, *we conclude that* $f : (-1, 1) \to \mathbb{R}$.

(b) *Suppose that* $f(x_1) = f(x_2)$*; i.e.,* $x_1, x_2 \in (-1, 1)$ *and* $\dfrac{x_1}{1-|x_1|} = \dfrac{x_2}{1-|x_2|}$. *Since* $f(x_1) = f(x_2)$ *we must have that* $f(x_1) < 0 \wedge f(x_2) < 0$ *or* $f(x_1) \geq 0 \wedge f(x_2) \geq 0$ *which implies that* $-1 < x_1 < 0 \wedge -1 < x_2 < 0$ *or* $1 > x_1 \geq 0 \wedge 1 > x_2 \geq 0$. *Now* $x_1, x_2 \in (-1, 0)$ *yields that* $f(x_1) = \dfrac{x_1}{1+x_1} = \dfrac{x_2}{1+x_2} = f(x_2)$, *while* $x_1, x_2 \in [0, 1)$ *leads to* $f(x_1) = \dfrac{x_1}{1-x_1} = \dfrac{x_2}{1-x_2} = f(x_2)$. *In either case, a simple calculation gives that* $x_1 = x_2$. *Since* x_1 *and* x_2 *were arbitrary,* $(\forall x_1)(\forall x_2)(f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$. *Therefore,* f *is one-to-one.*

(c) *Finally, fill in what is missing to finish showing that* f *is onto. Let* $w \in \mathbb{R}$. *Then either* $w < 0$ *or* $w \geq 0$. *For* $w < 0$, *let* $x = \dfrac{w}{1-w}$. *Then* $(1-w) > 0$ *and, because* $-1 < 0$, *we have that* $-1 + w < w$ *or* $-(1-w) < w$. *Hence,* $-1 < \dfrac{w}{1-w}$ *and we conclude the* $x \in \underset{(1)}{\underline{\hspace{3cm}}}$. *It follows that* $|x| = \underset{(2)}{\underline{\hspace{3cm}}}$ *and*

$$f(x) = \frac{x}{1-|x|} = \underset{(3)}{\underline{\hspace{8cm}}}.$$

For $w \geq 0$, *let* $x = \dfrac{w}{1+w}$. *Because* $1 > 0$ *and* $w \geq 0$ *implies that* $\underset{(4)}{\underline{\hspace{3cm}}} > w > 0$ *which is equivalent to having* $1 > x = \dfrac{w}{1+w} > 0$. *Hence,* $|x| = \underset{(5)}{\underline{\hspace{3cm}}}$ *and*

$$f(x) = \underset{(6)}{\underline{\hspace{8cm}}}.$$

Since $w \in \mathbb{R}$ *was arbitrary, we conclude that* f *maps* $(-1, 1)$ *onto* $\mathbb{R}$.

Acceptable responses are: (1) $(-1, 0)$, (2) $\dfrac{-w}{1-w}$, (3) $\left(\dfrac{w}{1-w}\right)\left(1 + \dfrac{w}{1-w}\right)^{-1} = w$, (4) $1 + w$, (5) $\dfrac{w}{1+w}$, (6) $\left(\dfrac{w}{1+w}\right)\left(1 - \dfrac{w}{1+w}\right)^{-1} = w$.

Given a relation from a set A to a set B, we saw two relations that could be used to describe or characterize properties of functions.

Definition 2.1.4 *Given sets A, B, and C, let $R \in \mathcal{P}(A \times B)$ and $S \in P(B \times C)$ where $\mathcal{P}(X)$ denotes the power set of X.*

1. *the **inverse** of R, denoted by R^{-1}, is $\{(y, x) : (x, y) \in R\}$;*
2. *the **composition** of R and S, denoted by $S \circ R$, is*
$$\{(x, z) \in A \times C : (\exists y)((x, y) \in R \wedge (y, z) \in S)\}.$$

Example 2.1.5 *For $R = \left\{(x, y) \in \mathbb{N} \times \mathbb{Z} : x^2 + y^2 \leq 4\right\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 2x + 1\}$, $R^{-1} = \{(0, 1), (-1, 1), (1, 1), (0, 2)\}$, $S^{-1} = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{x - 1}{2}\right\}$, and $S \circ R = \{(1, 1), (1, 3), (1, -1), (2, 1)\}$.*

Note that the inverse of a relation from a set A to a set B is always a relation from B to A; this is because a relation is an arbitrary subset of a Cartesian product that neither restricts nor requires any extent to which elements of A or B must be used. On the other hand, while the inverse of a function must be a relation, it need not be a function; even if the inverse is a function, it need not be a function with domain B. The following theorem, from MAT108, gave us necessary and sufficient conditions under which the inverse of a function is a function.

Theorem 2.1.6 *Let $f : A \to B$. Then f^{-1} is a function if and only if f is one–to–one. If f^{-1} is a function, then f^{-1} is a function from B into A if and only if f is a function from A onto B.*

We also saw many results that related inverses, compositions and the identity function. These should have included all or a large subset of the following.

Theorem 2.1.7 *Let $f : A \to B$ and $g : B \to C$. Then $g \circ f$ is a function from A into C.*

Theorem 2.1.8 *Suppose that A, B, C, and D are sets, $R \in \mathcal{P}(A \times B)$, $S \in \mathcal{P}(B \times C)$, and $T \in \mathcal{P}(C \times D)$. Then*

$$T \circ (S \circ R) = (T \circ S) \circ R.$$

and

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}.$$

Theorem 2.1.9 *Suppose that* A *and* B *are sets and that* $R \in \mathcal{P}(A \times B)$. *Then*

1. $R \circ R^{-1} \in \mathcal{P}(B \times B)$ *and, whenever* R *is single-valued,* $R \circ R^{-1} \subseteq I_B$
2. $R^{-1} \circ R \in \mathcal{P}(A \times A)$ *and, whenever* R *is one-to-one,* $R^{-1} \circ R \subseteq I_A$
3. $\left(R^{-1}\right)^{-1} = R$
4. $I_B \circ R = R$ *and* $R \circ I_A = R$.

Theorem 2.1.10 *For* $f : A \to B$ *and* $g : B \to C$,

1. *If* f *and* g *are one–to–one, then* $g \circ f$ *is one–to–one.*
2. *If* f *is onto* B *and* g *is onto* C, *then* $g \circ f$ *is onto* C.
3. *If* $g \circ f$ *is one–to–one, then* f *is one–to–one.*
4. *If* $g \circ f$ *is onto* C *then* g *is onto* C.

Theorem 2.1.11 *Suppose that* A, B, C, *and* D *are sets in the universe* $\mathcal{U}$.

1. *If* h *is a function having* $\operatorname{dom} h = A$, g *is a function such that* $\operatorname{dom} g = B$, *and* $A \cap B = \emptyset$, *then* $h \cup g$ *is a function such that* $\operatorname{dom}(h \cup g) = A \cup B$.
2. *If* $h : A \twoheadrightarrow C$, $g : B \twoheadrightarrow D$ *and* $A \cap B = \emptyset$, *then* $h \cup g : A \cup B \twoheadrightarrow C \cup D$.
3. *If* $h : A \overset{1-1}{\to} C$, $g : B \overset{1-1}{\to} D$, $A \cap B = \emptyset$, *and* $C \cap D = \emptyset$, *then*
$$h \cup g : A \cup B \overset{1-1}{\to} C \cup D.$$

Remark 2.1.12 *Theorem 2.1.11 can be used to give a slightly different proof of the result that was shown in Example 2.1.3. Notice that the relation* f *that was given in Example 2.1.3 can be realized as* $f_1 \cup f_2$ *where*

$$f_1 = \left\{\left(x, \frac{x}{1+x}\right) \in \mathbb{R} \times \mathbb{R} : -1 < x < 0\right\}$$

and

$$f_2 = \left\{\left(x, \frac{x}{1-x}\right) \in \mathbb{R} \times \mathbb{R} : 0 \le x < 1\right\};$$

for this set-up, we would show that $f_1 : (-1, 0) \overset{1-1}{\twoheadrightarrow} (-\infty, 0)$ *and* $f_2 : [0, 1) \overset{1-1}{\twoheadrightarrow} [0, \infty)$ *and claim* $f_1 \cup f_2 : (-1, 1) \overset{1-1}{\twoheadrightarrow} \mathbb{R}$ *from Theorem 2.1.11, parts (#2) and (#3).*

2.2 A Review of Cardinal Equivalence

Definition 2.2.1 *Two sets A and B are said to be* ***cardinally equivalent****, denoted by $A \sim B$, if and only if $(\exists f)\left(f : A \overset{1-1}{\twoheadrightarrow} B\right)$. If $A \sim B$ (read "A is equivalent to B"), then A and B are said to have the same cardinality.*

Example 2.2.2 *Let $A = \{0\}$ and $B = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Make use of the relation*

$$\{(x, f(x)) : x \in [0,1]\}$$

where

$$f(x) = \begin{cases} \frac{1}{2} & \text{, if } x \in A \\ \frac{x}{1+2x} & \text{, if } x \in B \\ x & \text{, if } x \in [0,1] - (A \cup B) \end{cases}$$

to prove that the closed interval $[0,1]$ is cardinally equivalent to the open interval $(0,1)$.

Proof. *Let $F = \{(x, f(x)) : x \in [0,1]\}$ where f is defined above. Then $F = \{(x, g_1) : x \in A \cup B\} \cup \{(x, g_2) : x \in ([0,1] - (A \cup B))\}$ where*

$$g_1(x) = \begin{cases} \frac{1}{2} & \text{, if } x \in A \\ \frac{x}{1+2x} & \text{, if } x \in B \end{cases} \quad \text{and} \quad g_2 = f \restriction_{[0,1]-(A\cup B)}.$$

Suppose that $x \in A \cup B$. Then either $x = 0$ or there exists $n \in \mathbb{N}$ such that $x = \frac{1}{n}$. It follows that $g_1(x) = g_1(0) = \frac{1}{2}$ or $g_1(x) = g_1\left(\frac{1}{n}\right) = \frac{\frac{1}{n}}{1+2\frac{1}{n}} = \frac{1}{n+2} \in (0,1)$. Since x was arbitrary, we have that

$$(\forall x)(x \in A \cup B \Rightarrow (\exists y)(y \in (0,1) \wedge g_1(x) = y)).$$

Thus, $\operatorname{dom}(g_1) = A \cup B$. *Furthermore, since* $(1 + 2 \cdot x) \neq 0$ *for* $x \in B$

$$\frac{x}{1+2x} = x \cdot (1 + 2 \cdot x)^{-1}$$

is defined and single-valued because $\cdot$ *and* $+$ *are the binary operations on the field* $\mathbb{R}$. *Hence,* $g_1 : A \cup B \to (0,1)$.

Since

$$g_1[A \cup B] = \left\{ \frac{1}{n+2} : n \in \mathbb{N} \right\} \underset{def}{=} C,$$

we have that $g_1 : A \cup B \twoheadrightarrow C$. *Now suppose that* $x_1, x_2 \in A \cup B$ *are such that* $g_1(x_1) = g_1(x_2)$. *Then either* $g_1(x_1) = g_1(x_2) = \frac{1}{2}$ *or* $g_1(x_1), g_1(x_2) \in \left\{ \frac{1}{n+2} : n \in \mathbb{N} \right\}$. *In the first case, we have that* $x_1 = x_2 = 0$. *In the second case, we have that* $g_1(x_1) = g_1(x_2) \Rightarrow$

$$\frac{x_1}{1+2x_1} = \frac{x_2}{1+2x_2} \Leftrightarrow x_1 + 2x_1x_2 = x_2 + 2x_2x_1 \Leftrightarrow x_1 = x_2.$$

Since x_1 *and* x_2 *were arbitrary,*

$$(\forall x_1, x_2)(x_1, x_2 \in A \cup B \wedge g_1(x_1) = g_1(x_2) \Rightarrow x_1 = x_2)\text{; i.e.,}$$

g_1 *is one-to-one. Therefore,*

$$g_1 : A \cup B \overset{1-1}{\twoheadrightarrow} C.$$

Note that $[0,1] - (A \cup B) = (0,1) - C$. *Thus,* g_2, *as the identity function on* $(0,1) - C$, *is one-to-one and onto. That is,*

$$g_2 : ((0,1) - C) \overset{1-1}{\twoheadrightarrow} ((0,1) - C).$$

From Theorem 2.1.11 (#2) *and* (#3), $g_1 : A \cup B \overset{1-1}{\twoheadrightarrow} C$, $g_2 : ((0,1) - C) \overset{1-1}{\twoheadrightarrow} ((0,1) - C)$, $([0,1] - (A \cup B)) \cap (A \cup B) = \emptyset$ *and* $((0,1) - C) \cap C = \emptyset$ *implies that*

$$g_1 \cup g_2 : (A \cup B) \cup ((0,1) - C) \overset{1-1}{\twoheadrightarrow} C \cup ((0,1) - C). \tag{*}$$

Substituting $((0,1)-C)=[0,1]-(A\cup B)$ *in addition to noting that*

$$(A\cup B)\cup([0,1]-(A\cup B))=[0,1]$$

and

$$C\cup((0,1)-C)=(0,1)\,,$$

we conclude from (*) that

$$F=g_1\cup g_2:[0,1]\overset{1-1}{\twoheadrightarrow}(0,1)\,.$$

Therefore, $|[0,1]|=|(0,1)|$. ■

For the purpose of describing and showing that sets are “finite”, we make use of the following collection of “master sets.” For each $k\in\mathbb{J}$, let

$$\mathbb{J}_k=\{j\in\mathbb{J}:1\le j\le k\}.$$

For $k\in\mathbb{J}$, the set $\mathbb{J}_k$ is defined to have cardinality k. The following definition offers a classification that distinguishes set sizes of interest.

Definition 2.2.3 *Let S be a set in the universe $\mathcal{U}$. Then*

1. *S is* ***finite*** $\Leftrightarrow((S=\emptyset)\vee(\exists k)\,(k\in\mathbb{J}\wedge S\sim\mathbb{J}_k)$.
2. *S is* ***infinite*** $\Leftrightarrow$ *S is not finite.*
3. *S is* ***countably infinite*** *or* ***denumerable*** $\Leftrightarrow S\sim\mathbb{J}$.
4. *S is* ***at most countable*** $\Leftrightarrow$ *((S is finite) $\vee$ (S is denumerable)).*
5. *S is* ***uncountable*** $\Leftrightarrow$ *S is neither finite nor countably infinite.*

Recall that if $S=\emptyset$, then it is said to have cardinal number 0, written $|S|=0$. If $S\sim\mathbb{J}_k$, then S is said to have cardinal number k; i.e., $|S|=k$.

Remark 2.2.4 *Notice that the term countable has been omitted from the list given in Definition 2.2.3; this was done to stress that the definition of countable given by the author of our textbook is different from the definition that was used in all the MAT108 sections. The term “at most countable” corresponds to what was defined as countable in MAT108. In these Companion Notes, we will avoid confusion by not using the term countable; when reading your text, keep in mind that Rudin uses the term countable for denumerable or countably infinite.*

We know an infinite set is one that is not finite. Now it would be nice to have some meaningful infinite sets. The first one we think of is $\mathbb{N}$ or $\mathbb{J}$. While this claim may seem obvious, it needs proving. This leads to the following

Proposition 2.2.5 *The set $\mathbb{J}$ is infinite.*

Space for comments.

Proof. Since $\{\emptyset\} \underset{def}{=} 1 \in \mathbb{J}$, $\mathbb{J}$ is not empty. To prove that $\neg\,(\exists k)\,(k \in \mathbb{J} \wedge \mathbb{J}_k \sim \mathbb{J})$ is suffices to show that $(\forall k)\,(\forall f)\left(\left(k \in \mathbb{J} \wedge f : \mathbb{J}_k \xrightarrow{1-1} \mathbb{J}\right) \Rightarrow f\,[\mathbb{J}_k] \neq \mathbb{J}\right)$. Suppose that $k \in \mathbb{J}$ and f is such that $f : \mathbb{J}_k \xrightarrow{1-1} \mathbb{J}$. Let $n = f(1) + f(2) + \cdots + f(k) + 1$. For each j, $1 \leq j \leq k$, we have that $f(j) > 0$. Hence, n is a natural number that is greater than each $f(j)$. Thus, $n \neq f(j)$ for any $j \in \mathbb{J}_k$. But then $n \notin \operatorname{rng}(f)$ from which we conclude that f is not onto $\mathbb{J}$; i.e., $f\,[\mathbb{J}_k] \neq \mathbb{J}$. Since k and f were arbitrary, we have that $(\forall k)\,(\forall f)\left(\left(k \in \mathbb{J} \wedge f : \mathbb{J}_k \xrightarrow{1-1} \mathbb{J}\right) \Rightarrow f\,[\mathbb{J}_k] \neq \mathbb{J}\right)$ which is equivalent to the claim that $(\forall k)\,(k \in \mathbb{J} \Rightarrow \mathbb{J}_k \nsim \mathbb{J})$. Because

$$((\mathbb{J} \neq \emptyset) \wedge \neg\,(\exists k)\,(k \in \mathbb{J} \wedge \mathbb{J} \sim \mathbb{J}_k),$$

it follows that $\mathbb{J}$ is not finite as claimed. ■

Remark 2.2.6 *From the Pigeonhole Principle (various forms of which were visited in MAT108), we know that, for any set X,*

$$X \textit{ finite } \Rightarrow (\forall Y)\,(Y \subset X \wedge Y \neq X \Rightarrow Y \nsim X)\,.$$

The contrapositive tautology yields that

$$\neg\,(\forall Y)\,(Y \subset X \wedge Y \neq X \Rightarrow Y \nsim X) \Rightarrow \neg\,(X \textit{ is finite})$$

which is equivalent to

$$(\exists Y)\,(Y \subset X \wedge Y \neq X \wedge Y \sim X) \Rightarrow X \textit{ is infinite.} \qquad (\triangle)$$

In fact, $(\triangle)$ *could have been used as a alternative definition of infinite set. To see how* $(\triangle)$ *can be used to prove that a set is infinite, note that*

$$\mathbb{J}_e = \{n \in \mathbb{J} : 2|n\}$$

is such that $\mathbb{J}_e \subsetneq \mathbb{J}$ *and* $\mathbb{J}_e \sim \mathbb{J}$ *where that latter follows because* $f(x) = 2x : \mathbb{J} \overset{1-1}{\twoheadrightarrow} \mathbb{J}_e$*; consequently,* $\mathbb{J}$ *is infinite.*

Recall that the cardinal number assigned to $\mathbb{J}$ is $\aleph_0$ which is read as "aleph naught." Also shown in MAT108 was that the set $\mathcal{P}(\mathbb{J})$ cannot be (cardinally) equivalent to $\mathbb{J}$; this was a special case of

Theorem 2.2.7 (Cantor's Theorem) *For any set* S, $|S| < |\mathcal{P}(S)|$.

Remark 2.2.8 *It can be shown, and in some sections of MAT108 it was shown, that* $\mathcal{P}(\mathbb{J}) \sim \mathbb{R}$*. Since* $|\mathbb{J}| < |\mathbb{R}|$*, the cardinality of* $\mathbb{R}$ *represents a different "level of infinite." The symbol given for the cardinality of* $\mathbb{R}$ *is* **c***, an abbreviation for continuum.*

Excursion 2.2.9 *As a memory refresher concerning proofs of cardinal equivalence, complete each of the following.*

1. *Prove* $(2, 4) \sim (-5, 20)$.

2. *Use the function* $f(n) = \begin{cases} \dfrac{n}{2} & , \quad n \in \mathbb{J} \wedge 2 \mid n \\[2ex] -\dfrac{n-1}{2} & , \quad n \in \mathbb{J} \wedge 2 \nmid n \end{cases}$

to prove that $\mathbb{Z}$ is denumerable.

*******For (1), one of the functions that would have worked is $f(x) = \frac{25}{2}x - 30$; justifying that $f : (2,4) \overset{1-1}{\twoheadrightarrow} (-5,20)$ involves only simple algebraic manipulations. Showing that the function given in (2) in one-to-one and onto involves applying elementary algebra to the several cases that need to be considered for members of the domain and range.*******

We close this section with a proposition that illustrates the general approach that can be used for drawing conclusions concerning the cardinality of the union of two sets having known cardinalities

Proposition 2.2.10 *The union a denumerable set and a finite set is denumerable; i.e.,*

$$(\forall A)\,(\forall B)\,(A \textit{ denumerable } \wedge\ B \textit{ finite } \Rightarrow (A \cup B) \textit{ is denumerable})\,.$$

Proof. Let A and B be sets such that A is denumerable and B is finite. First we will prove that $A \cup B$ is denumerable when $A \cap B = \emptyset$. Since B is finite, we have that either $B = \emptyset$ or there exists a natural number k and a function f such that $f : B \overset{1-1}{\twoheadrightarrow} \{j \in \mathbb{N} : j \leq k\}$.

If $B = \emptyset$, then $A \cup B = A$ is denumerable. If $B \neq \emptyset$, then let f be such that $f : B \overset{1-1}{\twoheadrightarrow} \mathbb{N}_k$ where $\mathbb{N}_k \underset{def}{=} \{j \in \mathbb{N}_k : j \leq k\}$. Since A is denumerable, there exists a function g such that $g : A \overset{1-1}{\twoheadrightarrow} \mathbb{N}$. Now let $h = \{(n, n+k) : n \in \mathbb{N}\}$. Because addition is a binary operation on $\mathbb{N}$ and $\mathbb{N}$ is closed under addition, for each $n \in \mathbb{N}$, $n + k$ is a uniquely determined natural number. Hence, we have that $h : \mathbb{N} \to \mathbb{N}$. Since $n \in \mathbb{N}$ implies that $n \geq 1$, from OF1, $n+k \geq 1+k$; consequently, $\{j \in \mathbb{N} : j \geq 1 + k\} = \mathbb{N} - \mathbb{N}_k$ is a codomain for h. Thus, $h : \mathbb{N} \to \mathbb{N} - \mathbb{N}_k$.

We will now show that h is one-to-one and onto $\mathbb{N} - \mathbb{N}_k$.

(i) Suppose that $h(n_1) = h(n_2)$; i.e., $n_1 + k = n_2 + k$. Since $\mathbb{N}$ is the set of natural numbers for the field of real numbers, there exists an additive inverse $(-k) \in \mathbb{R}$ such that $k + (-k) = (-k) + k = 0$. From associativity and substitution, we have that

$$\begin{aligned} n_1 = n_1 + (k + (-k)) &= (n_1 + k) + (-k) \\ &= (n_2 + k) + (-k) \\ &= n_2 + (k + (-k)) \\ &= n_2. \end{aligned}$$

Since n_1 and n_2 were arbitrary, $(\forall n_1)(\forall n_2)(h(n_1) = h(n_2) \Rightarrow n_1 = n_2)$; i.e., h is one-to-one.

(ii) Let $w \in \mathbb{N} - \mathbb{N}_k$. Then $w \in \mathbb{N}$ and $w \geq 1 + k$. By OF1, associativity of addition, and the additive inverse property,

$$w + (-k) \geq (1 + k) + (-k) = 1 + (k + (-k)) = 1.$$

Hence, $x \underset{def}{=} w + (-k) \in \mathbb{N} = \operatorname{dom}(h)$. Furthermore,

$$h(x) = x + k = (w + (-k)) + k = w + ((-k) + k) = w.$$

Since w was arbitrary, we have shown that

$$(\forall w)(w \in \mathbb{N} - \mathbb{N}_k \Rightarrow (\exists x)(x \in \mathbb{N} \wedge (x, w) \in h));$$

that is, h is onto.

From (i) and (ii), we conclude that $h : \mathbb{N} \overset{1-1}{\twoheadrightarrow} \mathbb{N} - \mathbb{N}_k$. From Theorem 2.1.10, parts (1) and (2), $g : A \overset{1-1}{\twoheadrightarrow} \mathbb{N}$ and $h : \mathbb{N} \overset{1-1}{\twoheadrightarrow} \mathbb{N} - \mathbb{N}_k$ implies that

$$h \circ g : A \overset{1-1}{\twoheadrightarrow} \mathbb{N} - \mathbb{N}_k.$$

Now we consider the new function $F = f \cup (h \circ g)$ from $B \cup A$ into $\mathbb{N}$ which can also be written as

$$F(x) = \begin{cases} f(x) & \text{for } x \in B \\ \\ (h \circ g)(x) & \text{for } x \in A \end{cases}.$$

Since $A \cap B = \emptyset$, $\mathbb{N} \cap (\mathbb{N} - \mathbb{N}_k) = \emptyset$, $f : B \overset{1-1}{\twoheadrightarrow} \mathbb{N}_k$ and $h \circ g : A \overset{1-1}{\twoheadrightarrow} \mathbb{N} - \mathbb{N}_k$, by Theorems 2.1.11, part (1) and (2), $F : B \cup A \overset{1-1}{\twoheadrightarrow} \mathbb{N} \cup (\mathbb{N} - \mathbb{N}_k) = \mathbb{N}$. Therefore, $B \cup A$ or $A \cup B$ is cardinally equivalent to $\mathbb{N}$; i.e., $A \cup B$ is denumerable.

If $A \cap B \neq \emptyset$, then we consider the sets $A - B$ and B. In this case, $(A - B) \cap B = \emptyset$ and $(A - B) \cup B = A \cup B$. Now the set B is finite and the set $A - B$ is denumerable. The latter follows from what we showed above because our proof for the function h was for k arbitrary, which yields that

$$(\forall k)\,(k \in \mathbb{N} \Rightarrow |\mathbb{N} - \mathbb{N}_k| = \aleph_0)\,.$$

From the argument above, we again conclude that $A \cup B = (A - B) \cup B$ is denumerable.

Since A and B were arbitrary,

$$(\forall A)\,(\forall B)\,(A \text{ denumerable } \wedge \ B \text{ finite } \Rightarrow (A \cup B) \text{ is denumerable})\,.$$

■

2.2.1 Denumerable Sets and Sequences

An important observation that we will use to prove some results concerning at most countable sets and families of such sets is the fact that a denumerable set can be “arranged in an (infinite) sequence.” First we will clarify what is meant by arranging a set as a sequence.

Definition 2.2.11 *Let A be a nonempty set. A **sequence** of elements of A is a function $f : \mathbb{J} \longrightarrow A$. Any $f : \mathbb{J}_k \longrightarrow A$ for a $k \in \mathbb{J}$ is a **finite sequence** of elements of A.*

For $f : \mathbb{J} \longrightarrow A$, letting $a_n = f(n)$ leads to the following common notations for the sequence: $\{a_n\}_{n=1}^{\infty}$, $\{a_n\}_{n \in \mathbb{J}}$, $\{a_n\}$, or $a_1, a_2, a_3, ..., a_n, ...$ It is important to notice the distinction between $\{a_n\}_{n=1}^{\infty}$ and $\{a_n : n \in \mathbb{J}\}$; the former is a sequence where the listed terms need not be distinct, while the latter is a set. For example, if $f : \mathbb{J} \longrightarrow \{1, 2, 3\}$ is the constant function $f(n) = 1$, then

$$\{a_n\}_{n=1}^{\infty} = 1, 1, 1, ...$$

while $\{a_n : n \in \mathbb{J}\} = \{1\}$.

Now, if A is a denumerable or countably infinite set then there exists a function g such that $g : \mathbb{J} \overset{1-1}{\twoheadrightarrow} A$. In this case, letting $g(n) = x_n$ leads to a sequence $\{x_n\}_{n \in \mathbb{J}}$ of elements of A that exhausts A; i.e., every element of A appears someplace in the sequence. This phenomenon explains our meaning to saying that the “elements of A can be arranged in an infinite sequence.” The proof of the following theorem illustrates an application of this phenomenon.

Theorem 2.2.12 *Every infinite subset of a countably infinite set is countably infinite.*

Proof. Let A be a denumerable set and E be an infinite subset of A. Because A is denumerable, it can be arranged in an infinite sequence, say $\{a_n\}_{n=1}^{\infty}$. Let

$$S_1 = \{m \in \mathbb{J} : a_m \in E\}.$$

Because S_1 is a nonempty set of natural numbers, by the Well-Ordering Principle, S_1 has a least element. Let n_1 denote the least element of S_1 and set

$$S_2 = \{m \in \mathbb{J} : a_m \in E\} - \{n_1\}.$$

Since E is infinite, S_2 is a nonempty set of natural numbers. By the Well-Ordering Principle, S_2 has a least element, say n_2. In general, for S_1, S_2, ..., S_{k-1} and $n_1, n_2, ..., n_{k-1}$, we choose

$$n_k = \min S_k \qquad where \qquad S_k = \{m \in \mathbb{J} : a_m \in E\} - \{n_1, n_2, ..., n_{k-1}\}$$

Use the space provided to convince yourself that this choice "arranges E into an infinite sequence $\left\{a_{n_k}\right\}_{k=1}^{\infty}$."

■

2.3 Review of Indexed Families of Sets

Recall that if $\mathcal{F}$ is an indexed family of subsets of a set S and Δ denotes the indexing set, then the **union of the sets in** $\mathcal{F} = \{A_\alpha : \alpha \in \Delta\}$, denoted by $\bigcup_{\alpha\in\Delta} A_\alpha$, is

$$\{p \in S : (\exists\beta)\left(\beta \in \Delta \wedge p \in A_\beta\right)\};$$

and the **intersection of the sets in** $\mathcal{F}$, denoted by $\bigcap_{\alpha\in\Delta} A_\alpha$, is

$$\{p \in S : (\forall\beta)\left(\beta \in \Delta \Rightarrow p \in A_\beta\right)\}.$$

Remark 2.3.1 *If $\mathcal{F}$ is a countably infinite or denumerable family of sets (subsets of a set S), then the indexing set is $\mathbb{J}$ or $\mathbb{N}$; in this case, the union and intersection over $\mathcal{F}$ are commonly written as $\bigcup_{j=1}^{\infty} A_j$ and $\bigcap_{j=1}^{\infty} A_j$, respectively. If $\mathcal{F}$ is a nonempty finite family of sets, then $\mathbb{J}_k$, for some $k \in \mathbb{J}$, can be used as an indexing set; in this case, the union and intersection over $\mathcal{F}$ are written as $\bigcup_{j=1}^{k} A_j$ and $\bigcap_{j=1}^{k} A_j$, respectively.*

It is important to keep in mind that, in an indexed family $\mathcal{F} = \{A_\alpha : \alpha \in \Delta\}$, different subscript assignments does not ensure that the sets represented are different. An example that you saw in MAT108 was with equivalence classes. For the relation $\equiv_3$that was defined over $\mathbb{Z}$ by $x \equiv_3 y \Leftrightarrow 3|\,(x-y)$, for any $\alpha \in \mathbb{Z}$, let $A_\alpha = [\alpha]_{\equiv_3}$; then $A_{-4} = A_2 = A_5$, though the subscripts are different. The set of equivalence classes from an equivalence relation do, however, form a pairwise disjoint family.

An indexed family $\mathcal{F} = \{A_\alpha : \alpha \in \Delta\}$ is **pairwise disjoint** if and only if

$$(\forall\alpha)\,(\forall\beta)\left(\alpha, \beta \in \Delta \wedge A_\alpha \cap A_\beta \neq \emptyset \Rightarrow A_\alpha = A_\beta\right);$$

it is **disjoint** if and only if $\bigcap_{\alpha\in\Delta} A_\alpha = \emptyset$. Note that being disjoint is a weaker condition that being pairwise disjoint.

Example 2.3.2 *For each $j \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers, let*

$$A_j = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : |x_1 - j| \leq 1 \wedge |x_2| \leq 1\},$$

Find $\bigcup_{j\in\mathbb{Z}} A_j$ and $\bigcap_{j\in\mathbb{Z}} A_j$.

Each A_j *consists of a "2 by 2 square" that is symmetric about the x-axis. For each* $j \in \mathbb{Z}$, A_j *and* A_{j+1} *overlap in the section where* $j \leq x_1 \leq j+1$, *while* A_j *and* A_{j+3} *have nothing in common. Consequently,* $\bigcup_{j\in\mathbb{Z}} A_j = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \leq 1\right\}$ *and* $\bigcap_{j\in\mathbb{Z}} A_j = \emptyset$.

Excursion 2.3.3 *For* $n \in \mathbb{N}$, *let* $C_n = \left[-3 + \frac{1}{2n}, \frac{5n + (-1)^n}{n}\right)$ *and* $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$. *Find* $\bigcup_{j\in\mathbb{N}} C_j$ *and* $\bigcap_{j\in\mathbb{N}} C_j$.

For this one, hopefully you looked at C_n for a few n. For example, $C_1 = \left[-\frac{5}{2}, 4\right)$, $C_2 = \left[-\frac{11}{4}, 5\frac{1}{2}\right)$, and $C_3 = \left[-\frac{17}{6}, 4\frac{2}{3}\right)$. Upon noting that the left endpoints of the intervals are decreasing to -3 while the right endpoints are oscillating above and below 5 and closing in on 5, we conclude that $\bigcup_{j\in\mathbb{N}} C_j = \left(-3, 5\frac{1}{2}\right)$ and $\bigcap_{j\in\mathbb{N}} C_j = \left[-\frac{5}{2}, 4\right)$.\

Excursion 2.3.4 *For* $j \in \mathbb{J}$, *let* $A_j = \left\{x \in \mathbb{R} : x \geq \sqrt{j}\right\}$. *Justify the claim that* $\mathcal{A} = \left\{A_j : j \in \mathbb{J}\right\}$ *is disjoint but not pairwise disjoint.*

*******Hopefully, your discussion led to your noticing that $A_k \cap A_m = A_{\max\{k,m\}}$. On the other hand, to justify that $\bigcap_{j\in\mathbb{N}} A_j = \emptyset$, you needed to note that given any fixed positive real number w there exists $p \in \mathbb{J}$ such that $w \notin A_p$; taking $p = \left\lfloor w^2 + 1 \right\rfloor$, where $\lfloor\bullet\rfloor$ denotes the greatest integer function, works.*******

2.4 Cardinality of Unions Over Families

We saw the following result, or a slight variation of it, in MAT108.

Lemma 2.4.1 *If A and B are disjoint finite sets, then $A \cup B$ is finite and*

$$|A \cup B| = |A| + |B| \,.$$

Excursion 2.4.2 *Fill in what is missing to complete the following proof of the Lemma.*

Space for Scratch Work

Proof. Suppose that A and B are finite sets such that $A \cap B = \emptyset$. If $A = \emptyset$ or $B = \emptyset$, then $A \cup B = \underset{(1)}{\underline{\hspace{1.5cm}}}$ or $A \cup B = \underset{(2)}{\underline{\hspace{1.5cm}}}$, respectively. In either case $A \cup B$ is $\underset{(3)}{\underline{\hspace{3.5cm}}}$, and $|\emptyset| = 0$ yields that $|A| + |B| = |A \cup B|$. If $A \neq \emptyset$ and $B \neq \emptyset$, then there exists $k, n \in \mathbb{N}$ such that $|A| = |\{i \in \mathbb{N} : i \le k\}|$ and $|B| = |\{i \in \mathbb{N} : i \le n\}|$. Hence there exist functions f and g such that $f : A \overset{1-1}{\twoheadrightarrow} \underset{(4)}{\underline{\hspace{2.5cm}}}$ and $g : \underset{(5)}{\underline{\hspace{4.5cm}}}$. Now let $H = \{k+1, k+2, \cdots, k+n\}$. Then the function $h(x) = k + x$ is such that $h : \{i \in \mathbb{N} : i \le n\} \overset{1-1}{\twoheadrightarrow} H$. Since the composition of one–to–one onto functions is a one–to–one correspondence, $F = h \circ g : \underset{(6)}{\underline{\hspace{6cm}}}$.

From Theorem 2.1.11, $A \cap B = \emptyset$,
$\{i \in \mathbb{N} : i \leq k\} \cap H = \emptyset$, $f : A \overset{1\text{–}1}{\twoheadrightarrow} \{i \in \mathbb{N} : i \leq k\}$, and
$F : B \overset{1\text{–}1}{\twoheadrightarrow} H$ implies that
$f \cup F : \underset{(7)}{\underline{\hspace{6cm}}}$. Since
$\{i \in \mathbb{N} : i \leq k\} \cup H = \underset{(8)}{\underline{\hspace{4.5cm}}}$, we
conclude that $A \cup B$ is $\underset{(9)}{\underline{\hspace{2cm}}}$ and
$|A \cup B| = \underset{(10)}{\underline{\hspace{2.5cm}}} = \underset{(11)}{\underline{\hspace{2.5cm}}}$. ■

*******Acceptable responses: (1) B, (2) A, (3) finite, (4) $\{i \in \mathbb{N} : i \leq k\}$ or A_k, (5) $B \overset{1\text{–}1}{\twoheadrightarrow} \{i \in \mathbb{N} : i \leq n\}$, (6) $B \overset{1\text{–}1}{\twoheadrightarrow} H$, (7) $A \cup B \overset{1\text{–}1}{\twoheadrightarrow} \{i \in \mathbb{N} : i \leq k\} \cup H$, (8) $\{i \in \mathbb{N} : i \leq k + n\}$ or A_{k+n}, (9) finite, (10) $k + n$, and (11) $|A| + |B|$.***

Lemma 2.4.1 and the Principle of Mathematical Induction can be used to prove

Theorem 2.4.3 *The union of a finite family of finite sets is finite.*

Proof. The proof is left as an exercise. ■

Now we want to extend the result of the theorem to a comparable result concerning denumerable sets. The proof should be reminiscent of the proof that $|\mathbb{Q}| = \aleph_0$.

Theorem 2.4.4 *The union of a denumerable family of denumerable sets is denumerable.*

Proof. For each $n \in \mathbb{J}$, let E_n be a denumerable set. Each E_n can be arranged as an infinite sequence, say $\left\{x_{nj}\right\}_{j=1}^{\infty}$. Then

$$\bigcup_{k \in \mathbb{J}} E_k = \left\{x_{nj} : n \in \mathbb{J} \wedge j \in \mathbb{J}\right\}.$$

Because E_1 is denumerable and $E_1 \subset \bigcup_{j \in \mathbb{J}} E_j$, we know that $\bigcup_{j \in \mathbb{J}} E_j$ is an infinite set. We can use the sequential arrangement to establish an infinite array; let the sequence corresponding to E_n form the nth row.

$$\begin{array}{cccccccc}
x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & \cdots & \cdots \\
x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & \cdots & \cdots \\
x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} & \cdots & \cdots \\
x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} & \cdots & \cdots \\
x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

.The terms in the infinite array can be rearranged in an expanding triangular array, such as

$$\begin{array}{cccccccc}
x_{11} & & & & & & & \\
x_{21} & x_{12} & & & & & & \\
x_{31} & x_{22} & x_{13} & & & & & \\
x_{41} & x_{32} & x_{23} & x_{14} & & & & \\
x_{51} & x_{42} & x_{33} & x_{24} & x_{15} & & & . \\
x_{61} & x_{52} & x_{43} & x_{34} & x_{25} & x_{16} & & \\
x_{71} & x_{62} & x_{53} & x_{44} & x_{35} & x_{26} & x_{17} & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots
\end{array}$$

This leads us to the following infinite sequence:

$$x_{11}, x_{21}, x_{21}, x_{31}, x_{22}, x_{13}, \ldots$$

Because we have not specified that each E_n is distinct, the infinite sequence may list elements from $\bigcup\limits_{k\in\mathbb{J}} E_k$ more than once; in this case, $\bigcup\limits_{k\in\mathbb{J}} E_k$ would correspond to an infinite subsequence of the given arrangement. Consequently, $\bigcup\limits_{k\in\mathbb{J}} E_k$ is denumerable, as needed. ■

Corollary 2.4.5 *If A is at most countable, and, for each $\alpha \in A$, B_α is at most countable, then*

$$T = \bigcup_{\alpha \in A} B_\alpha$$

is at most countable.

The last theorem in this section determines the cardinality of sets of $n - tuples$ that are formed from a given countably infinite set.

Theorem 2.4.6 *For A a denumerable set and $n \in \mathbb{J}$, let $T_n = \underbrace{A \times A \times \cdots \times A}_{n\,of\,them} = A^n$; i.e.,*

$$T_n = \left\{ (a_1, a_2, ..., a_n) : (\forall j) \left(j \in \mathbb{J} \wedge 1 \leq j \leq n \Rightarrow a_j \in A \right) \right\}.$$

Then T_n is denumerable.

Proof. Let $S = \{n \in \mathbb{J} : T_n \sim \mathbb{J}\}$. Since $T_1 = A$ and A is denumerable, $1 \in S$. Suppose that $k \in S$; i.e., $k \in \mathbb{J}$ and T_k is denumerable. Now $T_{k+1} = T_k \times A$ where it is understood that $((x_1, x_2, \ldots, x_k), a) = (x_1, x_2, \ldots, x_k, a)$. For each $b \in T_k$, $\{(b, a) : a \in A\} \sim A$. Hence,

$$(\forall b) \left(b \in T_k \Rightarrow \{(b, a) : a \in A\} \sim \mathbb{J} \right).$$

Because T_k is denumerable and

$$T_{k+1} = \bigcup_{b \in T_k} \{(b, a) : a \in A\}$$

it follows from Theorem 2.4.4 that T_{k+1} is denumerable; i.e., $(k+1) \in S$. Since k was arbitrary, we conclude that $(\forall k) (k \in S \Rightarrow (k+1) \in S)$.

By the Principle of Mathematical Induction,

$$1 \in S \wedge (\forall k) (k \in S \Rightarrow (k+1) \in S)$$

implies that $S = \mathbb{J}$. ■

Corollary 2.4.7 *The set of all rational numbers is denumerable.*

Proof. This follows immediately upon noting that

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z} \wedge q \in \mathbb{J} \wedge \gcd(p, q) = 1 \right\} \sim \{(p, q) \in \mathbb{Z} \times \mathbb{J} : \gcd(p, q) = 1\}$$

and $\mathbb{Z} \times \mathbb{J}$ is an infinite subset of $\mathbb{Z} \times \mathbb{Z}$ which is denumerable by the theorem. ■

2.5 The Uncountable Reals

In Example 2.1.3, it was shown that $f(x) = \dfrac{x}{1+|x|} : (-1,1) \overset{1-1}{\twoheadrightarrow} \mathbb{R}$. Hence, the interval $(-1,1)$ is cardinally equivalent to $\mathbb{R}$. The map $g(x) = \frac{1}{2}(x+1)$ can be used to show that $(-1,1) \sim (0,1)$. We noted earlier that $|\mathbb{J}| < |\mathbb{R}|$. For completeness, we restate the theorem and quickly review the proof.

Theorem 2.5.1 *The open interval* $(0,1)$ *is uncountable. Consequently,* $\mathbb{R}$ *is uncountable.*

Proof. Since $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} \subseteq (0,1)$ and $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} \sim \mathbb{J}$, we know that $(0,1)$ is not finite.

Suppose that

$$f : \mathbb{J} \overset{1-1}{\twoheadrightarrow} (0,1).$$

Then we can write

$$\begin{array}{lcl} f(1) & = & 0.a_{11}a_{12}a_{13}a_{14}\cdots\cdots\cdots\cdots \\ f(2) & = & 0.a_{21}a_{22}a_{23}a_{24}\cdots\cdots\cdots\cdots \\ f(3) & = & 0.a_{31}a_{32}a_{33}a_{34}\cdots\cdots\cdots\cdots \\ \vdots & & \\ \vdots & & \\ f(n) & = & 0.a_{n1}a_{n2}a_{n3}a_{n4}\cdots\cdots\cdots\cdots \\ \vdots & & \\ \vdots & & \end{array}$$

where $a_{km} \in \{0,1,2,3,4,5,6,7,8,9\}$. Because f is one-to-one, we know that, if .20000... is in the listing, then .199999... is not.

Finally, let $m = 0.b_1b_2b_3b_4\cdots$, where $b_j = \begin{cases} \langle \quad \rangle, & \text{if} \quad a_{jj} \neq \langle \quad \rangle \\ [\quad], & \text{if} \quad a_{jj} = \langle \quad \rangle \end{cases}$ (The substitutions for $\langle \bullet \rangle$.and $[\bullet]$ are yours to choose.). Now justify that there is no $q \in \mathbb{J}$

such that $f(q) = m$.

Hence, $(\exists m)(\forall k)(k \in J \Rightarrow f(k) \neq m)$; i.e., f is not onto.

Since f was arbitrary, we have shown that

$$(\forall f)(f : \mathbb{J} \to (0,1) \wedge f \text{ one-to-one} \Rightarrow f \text{ is not onto}).$$

Because $[(P \wedge Q) \Rightarrow \neg M]$ is logically equivalent to $[P \Rightarrow \neg(Q \wedge M)]$ and $\neg[P \Rightarrow Q]$ is equivalent to $[P \wedge \neg Q]$ for any propositions P, Q and M, we conclude that

$$\begin{aligned} &\left[(\forall f)(f : \mathbb{J} \to (0,1) \Rightarrow \neg(f \text{ one-to-one} \wedge f \text{ is onto}))\right] \\ \Leftrightarrow \quad &(\forall f)\neg(f : \mathbb{J} \to (0,1) \wedge f \text{ one-to-one} \wedge f \text{ is onto}); \end{aligned}$$

i.e., $\neg(\exists f)\left(f : \mathbb{J} \overset{1-1}{\twoheadrightarrow} (0,1)\right)$. Hence, the open interval $(0,1)$ is an infinite set that is not denumerable. ■

Corollary 2.5.2 *The set of sequences whose terms are the digits* 0 *and* 1 *is an uncountable set.*

2.6 Problem Set B

1. For each of the following relations, find R^{-1}.
   (a) $R = \{(1,3), (1,5), (5,7), (10,12)\}$
   (b) $R = \left\{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x^2\right\}$
   (c) $R = \{(a,b) \in A \times B : a|b\}$ where $A = \mathbb{J}$ and $B = \{j \in \mathbb{Z} : |j| \leq 6\}$
2. Prove that each of the following is one-to-one on its domain.
   (a) $f(x) = \dfrac{2x+5}{3x-2}$

   (b) $f(x) = x^3$

3. Prove that $f(x) = x^2 - 6x + 5$ maps $\mathbb{R}$ onto $[-4, \infty)$.

4. Prove each of the following parts of theorems that were stated in this chapter.

   (a) Suppose that A, B, C, and D are sets, $R \in \mathcal{P}(A \times B)$, $S \in \mathcal{P}(B \times C)$, and $T \in \mathcal{P}(C \times D)$. Then $T \circ (S \circ R) = (T \circ S) \circ R$

   (b) Suppose that A, B, and C are sets, $R \in \mathcal{P}(A \times B)$ and $S \in \mathcal{P}(B \times C)$. Then
   $$(S \circ R)^{-1} = R^{-1} \circ S^{-1}.$$

   (c) Suppose that A and B are sets and that $R \in \mathcal{P}(A \times B)$. Then
   $$R \circ R^{-1} \in \mathcal{P}(B \times B) \textit{ and, whenever } R \text{ is single-valued, } R \circ R^{-1} \subseteq I_B.$$

   (d) Suppose that A and B are sets and that $R \in \mathcal{P}(A \times B)$. Then
   $$R^{-1} \circ R \in \mathcal{P}(A \times A) \textit{ and, wnenever } R \text{ is one-to-one, } R^{-1} \circ R \subseteq I_A.$$

   (e) Suppose that A and B are sets and that $R \in \mathcal{P}(A \times B)$. Then
   $$\left(R^{-1}\right)^{-1} = R, I_B \circ R = Rand\ R \circ I_A = R.$$

5. For $f : A \to B$ and $g : B \to C$, prove each of the following.

   (a) If f and g are one–to–one, then $g \circ f$ is one–to–one.
   (b) If f is onto B and g is onto C, then $g \circ f$ is onto C.
   (c) If $g \circ f$ is one–to–one, then f is one–to–one.
   (d) If $g \circ f$ is onto C then g is onto C.

6. For A, B, C, and D sets in the universe $\mathcal{U}$, prove each of the following.

   (a) If h is a function having $\operatorname{dom} h = A$, g is a function such that $\operatorname{dom} g = B$, and $A \cap B = \emptyset$, then $h \cup g$ is a function such that $\operatorname{dom}(h \cup g) = A \cup B$.
   (b) If $h : A \twoheadrightarrow C$, $g : B \twoheadrightarrow D$ and $A \cap B = \emptyset$, then $h \cup g : A \cup B \twoheadrightarrow C \cup D$.

(c) If $h : A \overset{1-1}{\to} C$, $g : B \overset{1-1}{\to} D$, $A \cap B = \emptyset$, and $C \cap D = \emptyset$, then

$$h \cup g : A \cup B \overset{1-1}{\to} C \cup D.$$

7. Prove each of the following cardinal equivalences.
   - (a) $[-6, 10] \sim [1, 4]$
   - (b) $(-\infty, 3) \sim (1, \infty)$
   - (c) $(-\infty, 1) \sim (1, 2)$
   - (d) $\mathbb{W} = \mathbb{J} \cup \{0\} \sim \mathbb{Z}$
8. Prove that the set of natural numbers that are primes is infinite.
9. Let A be a nonempty finite set and B be a denumerable set. Prove that $A \times B$ is denumerable.
10. Find the union and intersection of each of the following families of sets.
    - (a) $\mathcal{A} = \{\{1, 3, 5\}, \{2, 3, 4, 5, 6\}, \{0, 3, 7, 9\}\}$
    - (b) $\mathcal{A} = \{A_n : n \in \mathbb{J}\}$ where $A_n = \left[\frac{1}{n}, 2 + \frac{1}{n}\right)$
    - (c) $\mathcal{B} = \{B_n : n \in \mathbb{J}\}$ where $B_n = \left(-\frac{1}{n}, n\right)$
    - (d) $\mathcal{C} = \{C_n : n \in \mathbb{J}\}$ where $C_n = \left\{x \in \mathbb{R} : 4 - \frac{3}{n} < x < 6 + \frac{2}{3n}\right\}$
11. Prove that the finite union of finite sets is finite.
12. For $\mathbb{W} = \mathbb{J} \cup \{0\}$, let $F : \mathbb{W} \times \mathbb{W} \longrightarrow \mathbb{W}$ be defined by

$$F(i, j) = j + \frac{k(k+1)}{2}$$

where $k = i + j$. Prove that F is a one-to-one correspondence.

13. Prove that $\mathbb{Q} \times \mathbb{Q}$ is denumerable.

Chapter 3

METRIC SPACES and SOME BASIC TOPOLOGY

Thus far, our focus has been on studying, reviewing, and/or developing an understanding and ability to make use of properties of $\mathbb{R} = \mathbb{R}^1$. The next goal is to generalize our work to $\mathbb{R}^n$ and, eventually, to study functions on $\mathbb{R}^n$.

3.1 Euclidean n-space

The set $\mathbb{R}^n$ is an extension of the concept of the Cartesian product of two sets that was studied in MAT108. For completeness, we include the following

Definition 3.1.1 *Let S and T be sets. The* ***Cartesian product*** *of S and T, denoted by $S \times T$, is*

$$\{(p, q) : p \in S \wedge q \in T\}.$$

The Cartesian product of any finite number of sets $S_1, S_2, ..., S_N$, denoted by $S_1 \times S_2 \times \cdots \times S_N$, is

$$\left\{(p_1, p_2, ..., p_N) : (\forall j)\left((j \in \mathbb{J} \wedge 1 \leq j \leq N) \Rightarrow p_j \in S_j\right)\right\}.$$

*The object $(p_1, p_2, ..., p_N)$ is called an N-**tuple**.*

Our primary interest is going to be the case where each set is the set of real numbers.

Definition 3.1.2 ***Real*** ***n-space***, *denoted* $\mathbb{R}^n$, *is the set all ordered n-tuples of real numbers; i.e.,*

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) : x_1, x_2, ..., x_n \in \mathbb{R}\}.$$

Thus, $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ of them}}$, *the Cartesian product of* $\mathbb{R}$ *with itself n times.*

Remark 3.1.3 *From MAT108, recall the definition of an* ***ordered pair****:*

$$(a, b) \underset{def}{=} \{\{a\}, \{a, b\}\}.$$

This definition leads to the more familiar statement that $(a, b) = (c, d)$ *if and only if* $a = b$ *and* $c = d$. *It also follows from the definition that, for sets A, B and C,* $(A \times B) \times C$ *is, in general, not equal to* $A \times (B \times C)$*; i.e., the Cartesian product is not associative. Hence, some conventions are introduced in order to give meaning to the extension of the binary operation to more that two sets. If we define ordered triples in terms of ordered pairs by setting* $(a, b, c) = ((a, b), c)$*; this would allow us to claim that* $(a, b, c) = (x, y, z)$ *if and only if* $a = x$, $b = y$, *and* $c = z$. *With this in mind, we interpret the Cartesian product of sets that are themselves Cartesian products as "big" Cartesian products with each entry in the tuple inheriting restrictions from the original sets. The point is to have helpful descriptions of objects that are described in terms of n-tuple.*

Addition and scalar multiplication on n-tuple is defined by

$$(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

and

$$\alpha\,(x_1, x_2, ..., x_n) = (\alpha x_1, \alpha x_2, ..., \alpha x_n)\,, \text{ for } \alpha \in \mathbb{R}, \text{ respectively.}$$

The geometric meaning of addition and scalar multiplication over $\mathbb{R}^2$ and $\mathbb{R}^3$ as well as other properties of these vector spaces was the subject of extensive study in vector calculus courses (MAT21D on this campus). For each n, $n \geq 2$, it can be shown that $\mathbb{R}^n$ is a real vector space.

Definition 3.1.4 *A* ***real vector space*** $\mathbb{V}$ *is a set of elements called* ***vectors****, with given operations of* ***vector addition*** $+ : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$ *and* ***scalar multiplication*** $\cdot : \mathbb{R} \times \mathbb{V} \longrightarrow \mathbb{V}$ *that satisfy each of the following:*

1. $(\forall \mathbf{v})\ (\forall \mathbf{w})\ (\mathbf{v}, \mathbf{w} \in \mathbb{V} \Rightarrow \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v})$ *commutativity*
2. $(\forall \mathbf{u})\ (\forall \mathbf{v})\ (\forall \mathbf{w})\ (\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V} \Rightarrow \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w})$ *associativity*
3. $(\exists \mathbf{0})\ (\mathbf{0} \in \mathbb{V} \wedge (\forall \mathbf{v})\ (\mathbf{v} \in \mathbb{V} \Rightarrow \mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}))$ *zero vector*
4. $(\forall \mathbf{v})\ (\mathbf{v} \in \mathbb{V} \Rightarrow\ (\exists\, (-\mathbf{v}))\ ((-\mathbf{v}) \in \mathbb{V} \wedge \mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}))$ *negatives*
5. $(\forall \lambda)\ (\forall \mathbf{v})\ (\forall \mathbf{w})\ (\lambda \in \mathbb{R} \wedge \mathbf{v}, \mathbf{w} \in \mathbb{V} \Rightarrow \lambda \cdot (\mathbf{v} + \mathbf{w}) = \lambda \cdot \mathbf{v} + \lambda \cdot \mathbf{w})$ *distributivity*
6. $(\forall \lambda)\ (\forall \gamma)\ (\forall \mathbf{w})\ (\lambda, \gamma \in \mathbb{R} \wedge \mathbf{w} \in \mathbb{V} \Rightarrow \lambda\, (\gamma \cdot \mathbf{w}) = (\lambda \gamma) \cdot \mathbf{w})$ *associativity*
7. $(\forall \lambda)\ (\forall \gamma)\ (\forall \mathbf{w})\ (\lambda, \gamma \in \mathbb{R} \wedge \mathbf{w} \in \mathbb{V} \Rightarrow\ (\lambda + \gamma) \cdot \mathbf{w} = \lambda \cdot \mathbf{w} + \gamma \cdot \mathbf{w})$ *distributivity*
8. $(\forall \mathbf{v})\ (\mathbf{v} \in \mathbb{V} \Rightarrow 1 \cdot \mathbf{v} = \mathbf{v} \cdot 1 = \mathbf{v})$ *multiplicative identity*

Given two vectors, $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$, the **inner product** (also known as the scalar product) is

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{n} x_j y_j ;$$

and the **Euclidean norm** (or magnitude) of $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is given by

$$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{j=1}^{n} \left(x_j\right)^2}.$$

The vector space $\mathbb{R}^n$ together with the inner product and Euclidean norm is called **Euclidean** n-space. The following two theorems pull together the basic properties that are satisfied by the Euclidean norm.

Theorem 3.1.5 *Suppose that* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$. *Then*

(a) $|\mathbf{x}| \geq 0$;

(b) $|\mathbf{x}| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$;

(c) $|\alpha \mathbf{x}| = |\alpha|\, |\mathbf{x}|$; *and* *(d)* $|\mathbf{x}\cdot\mathbf{y}| \le |\mathbf{x}|\,|\mathbf{y}|$.

Excursion 3.1.6 *Use Schwarz's Inequality to justify part (d). For* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ *in* $\mathbb{R}^n$,

$|\mathbf{x}\cdot\mathbf{y}|^2 =$

Remark 3.1.7 *It often helps to take our observations back to the setting that is "once removed" from* $\mathbb{R}^1$. *For the case* $\mathbb{R}^2$*, the statement given in part (d) of the theorem relates to the dot product of two vectors: For* $\xi = \overrightarrow{(x_1, x_2)}$ *and* $\eta = \overrightarrow{(y_1, y_2)}$*, we have that*

$$\xi\cdot\eta = x_1y_1 + x_2y_2$$

which, in vector calculus, was shown to be equivalent to $|\xi||\eta|cos\theta$ *where* θ *is the angle between the vectors* ξ *and* η.

Theorem 3.1.8 (The Triangular Inequalities) *Suppose that* $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ *and* $\mathbf{z} = (z_1, z_2, \ldots, z_N)$ *are elements of* $\mathbb{R}^N$. *Then*

(a) $|\mathbf{x}+\mathbf{y}| \le |\mathbf{x}| + |\mathbf{y}|$*; i.e.,*

$$\left(\sum_{j=1}^{N}(x_j + y_j)^2\right)^{1/2} \le \left(\sum_{j=1}^{N}x_j^2\right)^{1/2} + \left(\sum_{j=1}^{N}y_j^2\right)^{1/2}$$

where $(\cdots)^{1/2}$ *denotes the positive square root and equality holds if and only if either all the* x_j *are zero or there is a nonnegative real number* λ *such that* $y_j = \lambda x_j$ *for each* j, $1 \le j \le N$*; and*

(b) $|\mathbf{x}-\mathbf{z}| \le |\mathbf{x}-\mathbf{y}| + |\mathbf{y}-\mathbf{z}|$*; i.e.,*

$$\left(\sum_{j=1}^{N}(x_j - z_j)^2\right)^{1/2} \le \left(\sum_{j=1}^{N}(x_j - y_j)^2\right)^{1/2} + \left(\sum_{j=1}^{N}(y_j - z_j)^2\right)^{1/2}$$

where $(\cdots)^{1/2}$ *denotes the positive square root and equality holds if and only if there is a real number* r*, with* $0 \le r \le 1$*, such that* $y_j = rx_j + (1-r)z_j$ *for each* j*,* $1 \le j \le N$*.*

Remark 3.1.9 *Again, it is useful to view the triangular inequalities on "familiar ground." Let* $\xi = \overrightarrow{(x_1, x_2)}$ *and* $\eta = \overrightarrow{(y_1, y_2)}$*. Then the inequalities given in Theorem 3.1.8 correspond to the statements that were given for the complex numbers; i.e., statements concerning the lengths of the vectors that form the triangles that are associated with finding* $\xi + \eta$ *and* $\xi - \eta$*.*

Observe that, for $C = \{(x,y) : x^2 + y^2 = 1\}$ and $I = \{x : a \le x \le b\}$ where $a < b$, the Cartesian product of the circle C with I, $C \times I$, is the right circular cylinder,

$$U = \{(x,y,z) : x^2 + y^2 = 1 \wedge a \le z \le b\},$$

and the Cartesian product of I with C, $I \times C$, is the right circular cylinder,

$$V = \{(x,y,z) : a \le x \le b, y^2 + z^2 = 1\}.$$

If graphed on the same $\mathbb{R}^3$-coordinate system, U and V are different objects due to different orientation; on the other hand, U and V have the same height and radius which yield the same volume, surface area; etc. Consequently, distinguishing U from V depends on perspective and reason for study. In the next section, we lay the foundation for properties that place U and V in the same category.

3.2 Metric Spaces

In the study of $\mathbb{R}^1$ and functions on $\mathbb{R}^1$ the length of intervals and intervals to describe set properties are useful tools. Our starting point for describing properties for sets in $\mathbb{R}^n$ is with a formulation of a generalization of distance. It should come as no surprise that the generalization leads us to multiple interpretations.

Definition 3.2.1 *Let* S *be a set and suppose that* $d : S \times S \longrightarrow \mathbb{R}^1$*. Then* d *is said to be a* ***metric (distance function)*** *on* S *if and only if it satisfies the following three properties:*

(i) $(\forall x)\,(\forall y)\left[(x,y) \in S \times S \Rightarrow d(x,y) \ge 0 \wedge (d(x,y) = 0 \Leftrightarrow x = y)\right]$,

(ii) $(\forall x)(\forall y)\left[(x, y) \in S \times S \Rightarrow d(y, x) = d(x, y)\right]$ *(symmetry), and*

(iii) $(\forall x)(\forall y)(\forall z)\left[x, y, z \in S \Rightarrow d(x, z) \leq d(x, y) + d(y, z)\right]$ *(triangle inequality).*

Definition 3.2.2 *A **metric space** consists of a pair* (S, d)*–a set, S, and a metric, d, on S.*

Remark 3.2.3 *There are three commonly used (studied) metrics for the set* $\mathbb{R}^N$*. For* $\mathbf{x} = (x_1, x_2, ..., x_N)$ *and* $\mathbf{y} = (y_1, y_2, ..., y_N)$*, we have:*

- $(\mathbb{R}^N, d)$ *where* $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{j=1}^{N} \left(x_j - y_j\right)^2}$*, the Euclidean metric,*
- $(\mathbb{R}^N, D)$ *where* $D(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{N} |x_j - y_j|$*, and*
- $(\mathbb{R}^N, d_\infty)$ *where* $d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \leq j \leq N} \left|x_j - y_j\right|$.

Proving that d, D, and d_∞ are metrics is left as an exercise.

Excursion 3.2.4 *Graph each of the following on Cartesian coordinate systems*

1. $A = \{x \in \mathbb{R}^2 : d(\mathbf{0}, \mathbf{x}) \leq 1\}$

2. $B = \{x \in \mathbb{R}^2 : D(\mathbf{0}, \mathbf{x}) \leq 1\}$

3. $C = \{x \in \mathbb{R}^2 : d_\infty(\mathbf{0}, \mathbf{x}) \le 1\}$

*******For (1), you should have gotten the closed circle with center at origin and radius one; for (2), your work should have led you to a "diamond" having vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$; the closed shape for (3) is the square with vertices $(1, -1)$, $(1, 1)$, $(-1, 1)$, and $(-1, -1)$.*******

Though we haven't defined continuous and integrable functions yet as a part of this course, we offer the following observation to make the point that metric spaces can be over different objects. Let $\mathcal{C}$ be the set of all functions that are continuous real valued functions on the interval $I = (x : 0 < x \le 1\}$. Then there are two natural metrics to consider on the set $\mathcal{C}$; namely, for f and g in $\mathcal{C}$ we have

(1) $(\mathcal{C}, d)$ where $d(f, g) = \max\limits_{0 \le x \le 1} |f(x) - g(x)|$, and

(2) $(\mathcal{C}, \overline{d})$ where $\overline{d}(f, g) = \int_0^1 |f(x) - g(x)| dx$.

Because metrics on the same set can be distinctly different, we would like to distinguish those that are related to each other in terms of being able to "travel between" information given by them. With this in mind, we introduce the notion of equivalent metrics.

Definition 3.2.5 *Given a set S and two metric spaces (S, d_1) and (S, d_2), d_1 and d_2 are said to be **equivalent metrics** if and only if there are positive constants c and C such that $cd_1(x, y) \le d_2(x, y) \le Cd_1(x, y)$ for all x, y in S.*

Excursion 3.2.6 *As the result of one of the Exercises in Problem Set C, you will know that the metrics d and d_∞ on $\mathbb{R}^2$ satisfy $d_\infty(\mathbf{x}, \mathbf{y}) \le d(\mathbf{x}, \mathbf{y}) \le \sqrt{2} \cdot d_\infty(\mathbf{x}, \mathbf{y})$.*

1. *Let $A = \{x \in \mathbb{R}^2 : d(0, x) \le 1\}$. Draw a figure showing the boundary of A and then show the largest circumscribed square that is symmetric about*

the origin and the square, symmetric about the origin, that circumscribes the boundary of A.

2. *Let* $C = \{x \in \mathbb{R}^2 : d_\infty(0, x) \le 1\}$. *Draw a figure showing the boundary of* C *and then show the largest circumscribed circle that is centered at the origin and the circle, centered at the origin, that circumscribes the boundary of* C.

*******For (1), your outer square should have corresponded to $\left\{\mathbf{x} = (x_1 x_2) \in \mathbb{R}^2 : d_\infty(\mathbf{0}, \mathbf{x}) = \sqrt{2}\right\}$; the outer circle that you showed for part of (2) should have corresponded to $\left\{\mathbf{x} = (x_1 x_2) \in \mathbb{R}^2 : d(\mathbf{0}, \mathbf{x}) = \sqrt{2}\right\}$.*******

Excursion 3.2.7 *Let* $E = \{(cos\theta, sin\theta) : 0 \le \theta < 2\pi\}$ *and define* $d^*(p_1, p_2) = |\theta_1 - \theta_2|$ *where* $p_1 = (cos\theta_1, sin\theta_1)$ *and* $p_2 = (cos\theta_2, sin\theta_2)$. *Show that* (E, d^*) *is* *a metric space.*

The author of our textbook refers to an open interval $(a,b)=\{x\in\mathbb{R}:a<x<b\}$ as a **segment** which allows the term **interval** to be reserved for a closed interval $[a,b]=\{x\in\mathbb{R}:a\le x\le b\}$; half-open intervals are then in the form of $[a,b)$ or $(a,b]$.

Definition 3.2.8 *Given real numbers $a_1,a_2,...,a_n$ and $b_1,b_2,...,b_n$ such that $a_j<b_j$ for $j=1,2,...,n$,*

$$\left\{(x_1,x_2,...,x_n)\in\mathbb{R}^n:(\forall j)\left(1\le j\le n\Rightarrow a_j\le x_j\le b_j\right)\right\}$$

*is called an n-**cell**.*

Remark 3.2.9 *With this terminology, a 1-cell is an interval and a 2-cell is a rectangle.*

Definition 3.2.10 *If $\mathbf{x}\in\mathbb{R}^n$ and r is a positive real number, then the **open ball** with center $\mathbf{x}$ and radius r is given by*

$$B(\mathbf{x},r)=\left\{\mathbf{y}\in\mathbb{R}^n:|\mathbf{x}-\mathbf{y}|<r\right\};$$

and the **closed ball** *with center* x *and radius* r *is given by*

$$\overline{B(\mathbf{x},r)}=\left\{\mathbf{y}\in\mathbb{R}^n:|\mathbf{x}-\mathbf{y}|\leq r\right\}.$$

Definition 3.2.11 *A subset* E *of* $\mathbb{R}^n$ *is* ***convex*** *if and only if*

$$(\forall\mathbf{x})(\forall\mathbf{y})(\forall\lambda)\left[\mathbf{x},\mathbf{y}\in E\wedge 0<\lambda<1\Rightarrow\lambda\mathbf{x}+(1-\lambda)\mathbf{y}\in E\right]$$

Example 3.2.12 *For* $\mathbf{x}\in\mathbb{R}^n$ *and* r *a positive real number, suppose that* $\mathbf{y}$ *and* $\mathbf{z}$ *are in* $B(\mathbf{x},r)$. *If* λ *real is such that* $0<\lambda<1$, *then*

$$\begin{aligned}|\lambda\mathbf{y}+(1-\lambda)\mathbf{z}-\mathbf{x}| &= |\lambda(\mathbf{y}-\mathbf{x})+(1-\lambda)(\mathbf{z}-\mathbf{x})|\\ &\leq \lambda|\mathbf{y}-\mathbf{x}|+(1-\lambda)|\mathbf{z}-\mathbf{x}|\\ &< \lambda r+(1-\lambda)r=r\quad.\end{aligned}$$

Hence, $\lambda\mathbf{y}+(1-\lambda)\mathbf{z}\in B(\mathbf{x},r)$. *Since* y *and* z *were arbitrary,*

$$(\forall\mathbf{y})(\forall\mathbf{z})(\forall\lambda)\left[\mathbf{y},\mathbf{z}\in B(\mathbf{x},r)\wedge 0<\lambda<1\Rightarrow\lambda\mathbf{y}+(1-\lambda)\mathbf{z}\in B(\mathbf{x},r)\right];$$

that is, $B(\mathbf{x},r)$ *is a convex subset of* $\mathbb{R}^n$.

3.3 Point Set Topology on Metric Spaces

Once we have a distance function on a set, we can talk about the proximity of points. The idea of a segment (interval) in $\mathbb{R}^1$ is replaced by the concept of a neighborhood (closed neighborhood). We have the following

Definition 3.3.1 *Let* p_0 *be an element of a metric space* S *whose metric is denoted by* d *and* r *be any positive real number. The* ***neighborhood of the point*** p_0 *with radius* r *is denoted by* $N(p_0,r)$ *or* $N_r(p_0)$ *and is given by*

$$N_r(p_0)=\{p\in S:d(p,p_0)<r\}.$$

The closed neighborhood with center p_0 *and radius* r *is denoted by* $\overline{N_r(p_0)}$ *and is given by*

$$\overline{N_r(p_0)}=\{p\in S:d(p,p_0)\leq r\}.$$

Remark 3.3.2 *The sets A, B and C defined in Excursion 3.2.4 are examples of closed neighborhoods in $\mathbb{R}^2$ that are centered at $(0,0)$ with unit radius.*

What does the unit neighborhood look like for $(\mathbb{R}^2, \hat{d})$ where

$$\hat{d}(x,y) = \begin{cases} 0, & \text{if} \quad x = y \\ \\ 1, & \text{if} \quad x \neq y \end{cases} \quad \text{is known as the discrete metric?}$$

We want to use the concept of neighborhood to describe the nature of points that are included in or excluded from sets in relationship to other points that are in the metric space.

Definition 3.3.3 *Let A be a set in a metric space (S, d).*

1. *Suppose that p_0 is an element of A. We say the p_0 is an* ***isolated point*** *of A if and only if*

$$(\exists N_r(p_0))\left[N_r(p_0) \cap A = \{p_0\}\right]$$

2. *A point p_0 is a* ***limit point*** *of the set A if and only if*

$$(\forall N_r(p_0))\,(\exists p)\left[p \neq p_0 \wedge p \in A \cap N_r(p_0)\right].$$

*(**N.B.** A limit point need not be in the set for which it is a limit point.)*

3. *The set A is said to be* ***closed*** *if and only if A contains all of its limit points.*

4. *A point p is an* ***interior point*** *of A if and only if*

$$\left(\exists N_{r_p}(p)\right)\left[N_{r_p}(p) \subset A\right]$$

5. *The set* A *is* ***open*** *if and only if*

$$(\forall p)\left(p \in A \Rightarrow \left(\exists N_{r_p}(p)\right)\left[N_{r_p}(p) \subset A\right]\right);$$

   i.e., every point in A *is an interior point of* A*.*

Example 3.3.4 *For each of the following subsets of* $\mathbb{R}^2$ *use the space that is provided to justify the claims that are made for the given set.*

(a) $\{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \in \mathbb{J} \wedge |x_1 + x_2| < 5\}$ *is closed because is contains all none of its limit points.*

(b) $\{(x_1, x_2) \in \mathbb{R}^2 : 4 < x_1^2 \wedge x_2 \in \mathbb{J}\}$ *is neither open not closed.*

(c) $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 > |x_1|\}$ *is open.*

Our next result relates neighborhoods to the “open” and “closed” adjectives.

Theorem 3.3.5 *(a) Every neighborhood is an open set.*
(b) Every closed neighborhood is a closed set.

Use this space to draw some helpful pictures related to proving the results.

Proof. (a) Let $N_r(p_0)$ be a neighborhood. Suppose that $q \in N_r(p_0)$ and set $r_1 = d(p_0, q)$. Let $\rho = \frac{r - r_1}{4}$. If $x \in N_\rho(q)$, then $d(x, q) < \frac{r - r_1}{4}$ and the triangular inequality yields that

$$d(p_0, x) \le d(p_0, q) + d(q, x) < r_1 + \frac{r - r_1}{4} = \frac{3r_1 + r}{4} < r.$$

Hence, $x \in N_r(p_0)$. Since x was arbitrary, we conclude that

$$(\forall x)\left(x \in N_\rho(q) \Rightarrow x \in N_r(p_0)\right);$$

i.e., $N_\rho(q) \subset N_r(p_0)$.Therefore, q is an interior point of $N_r(p_0)$. Because q was arbitrary, we have that each element of $N_r(p_0)$ is an interior point. Thus, $N_r(p_0)$ is open, as claimed.

Excursion 3.3.6 *Fill in what is missing in order to complete the following proof of (b)*

Let $\overline{N_r(p_0)}$ be a closed neighborhood and suppose that q is a limit point of $\overline{N_r(p_0)}$. Then, for each $r_n = \frac{1}{n}$, $n \in \mathbb{J}$, there exists $p_n \neq q$ such that $p_n \in \overline{N_r(p_0)}$ and $d(q, p_n) < \frac{1}{n}$. Because $p_n \in \overline{N_r(p_0)}$, $d(p_0, p_n) \le r$ for each $n \in \mathbb{J}$. Hence, by the triangular inequality

$$d(q, p_0) \le d(q, p_n) + \underbrace{\qquad\qquad}_{(1)} \le \underbrace{\qquad\qquad}_{(2)}.$$

Since q and p_0 are fixed and $\frac{1}{n}$ goes to 0 as n goes to infinity, it follows that $d(q, p_0) \le r$; that is, $q \in \underbrace{\qquad\qquad}_{(3)}$. Finally, q and arbitrary limit point of $\overline{N_r(p_0)}$ leads to the conclusion that $\overline{N_r(p_0)}$ contains $\underbrace{\qquad\qquad\qquad\qquad}_{(4)}$. Therefore, $\overline{N_r(p_0)}$ is closed.

■

Acceptable responses are: (1) $d(p_n, p_0)$, (2) $\frac{1}{n} + r$, (3) $\overline{N_r(p_0)}$, (4) all of its limit points.

The definition of limit point leads us directly to the conclusion that only infinite subsets of metric spaces will have limit points.

Theorem 3.3.7 *Suppose that (X, d) is a metric space and $A \subset X$. If p is a limit point of A, then every neighborhood of p contains infinitely many points of A.*

Space for scratch work.

Proof. For a metric space (X, d) and $A \subset X$, suppose that $p \in X$ is such that there exists a neighborhood of p, $N(p)$, with the property that $N(p) \cap A$

is a finite set. If $N(p) \cap A = \emptyset$ or , $N(p) \cap A = \{p\}$, then p is not a limit point. Otherwise, $N(p) \cap A$ being finite implies that it can be realized as a finite sequence, say $q_1, q_2, , q_3, ..., q_n$ for some fixed $n \in \mathbb{J}$. For each j, $1 \le j \le n$, let $r_j = d\left(x, q_j\right)$. Set $\rho = \min\limits_{\substack{1 \le j \le n \\ q_j \ne p}} d\left(x, q_j\right)$. If $p \in \{q_1, q_2, , q_3, ..., q_n\}$, then $N_\rho(p) \cap A = \{p\}$; otherwise $N_\rho(p) \cap A = \emptyset$. In either case, we conclude that p is not a limit point of A.

We have shown that if $p \in X$ has a neighborhood, $N(p)$, with the property that $N(p) \cap A$ is a finite set, then p is not a limit point of $A \subset X$. From the contrapositive tautology it follows immediately that if p is a limit point of $A \subset X$, then every neighborhood of p contains infinitely many points of A. ■

Corollary 3.3.8 *Any finite subset of a metric space has no limit point.*

From the Corollary, we note that every finite subset of a metric space is closed because it contains all none of its limit points.

3.3.1 Complements and Families of Subsets of Metric Spaces

Given a family of subsets of a metric space, it is natural to wonder about whether or not the properties of being open or closed are passed on to the union or intersection. We have already seen that these properties are not necessarily transmitted when we look as families of subsets of $\mathbb{R}$.

Example 3.3.9 *Let* $\mathcal{A} = \{A_n : n \in \mathbb{J}\}$ *where* $A_n = \left[\frac{-3n+2}{n}, \frac{2n^2-n}{n^2}\right]$. *Note that* $A_1 = [-1, 1]$, $A_2 = \left[-2, \frac{3}{2}\right]$, *and* $A_3 = \left[-3 + \frac{2}{3}, 2 - \frac{1}{3}\right]$. *More careful inspection reveals that* $\frac{-3n+2}{n} = -3 + \frac{2}{n}$ *is strictly decreasing to* -3 *and* $n \to \infty$, $\frac{2n^2-n}{n^2} = 2 - \frac{1}{n}$ *is strictly increasing to* 2 *as* $n \to \infty$, *and* $A_1 = [-1, 1] \subset A_n$ *for each* $n \in \mathbb{J}$. *It follows that* $\bigcup\limits_{n \in \mathbb{J}} A_n = (-3, 2)$ *and* $\bigcap\limits_{n \in \mathbb{J}} A_n = A_1 = [-1, 1]$.

The example tells us that we may need some special conditions in order to claim preservation of being open or closed when taking unions and/or intersections over families of sets.

The other set operation that is commonly studied is complement or relative complement. We know that the complement of a segment in $\mathbb{R}^1$ is closed. This motivates us to consider complements of subsets of metric spaces in general. Recall the following

Definition 3.3.10 *Suppose that A and B are subsets of a set S. Then the* ***set difference*** *(or* ***relative complement****) $A - B$, read "A not B", is given by*

$$A - B = \{p \in S : p \in A \wedge p \notin B\};$$

the ***complement*** *of A, denoted by A^c, is $S - A$.*

Excursion 3.3.11 *Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ and*

$$B = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - 1| \leq 1 \wedge |x_2 - 1| \leq 1\}.$$

On separate copies of Cartesian coordinate systems, show the sets $A - B$ and $A^c = \mathbb{R}^2 - A$.

The following identities, which were proved in MAT108, are helpful when we are looking at complements of unions and intersections. Namely, we have

Theorem 3.3.12 (deMorgan's Laws) *Suppose that S is any space and $\mathcal{F}$ is a family of subsets of S. Then*

$$\left[\bigcup_{A\in\mathcal{F}} A\right]^c = \bigcap_{A\in\mathcal{F}} A^c$$

and

$$\left[\bigcap_{A\in\mathcal{F}} (A)\right]^c = \bigcup_{A\in\mathcal{F}} A^c.$$

The following theorem pulls together basic statement concerning how unions, intersections and complements effect the properties of being open or closed. Because their proofs are straightforward applications of the definitions, most are left as exercises.

Theorem 3.3.13 *Let S be a metric space.*

1. *The union of any family $\mathcal{F}$ of open subsets of S is open.*
2. *If $A_1, A_2, ..., A_m$ is a finite family of open subsets of S, then the intersection $\bigcap_{j=1}^{m} A_j$ is open.*
3. *For any subset A of S, A is closed if and only if A^c is open.*
4. *The intersection of any family $\mathcal{F}$ of closed subsets of S is closed.*
5. *If $A_1, A_2, ..., A_m$ is a finite family of closed subsets of S, then the union $\bigcup_{j=1}^{m} A_j$ is closed.*
6. *The space S is both open and closed.*
7. *The null set is both open and closed.*

Proof. (of #2) Suppose that $A_1, A_2, ..., A_m$ is a finite family of open subsets of S, and $x \in \bigcap_{j=1}^{m} A_j$. From $x \in \bigcap_{j=1}^{m} A_j$, it follows that $x \in A_j$ for each j, $1 \leq j \leq m$. Since each A_j is open, for each j, $1 \leq j \leq m$, there exists $r_j > 0$ such that $N_{r_j}(x) \subset A_j$. Let $\rho = \min_{1 \leq j \leq m} r_j$. Because $N_\rho(x) \subset A_j$ for each j, $1 \leq j \leq m$, we conclude that $N_\rho(x) \subset \bigcap_{j=1}^{m} A_j$. Hence, x is an interior point of $\bigcap_{j=1}^{m} A_j$. Finally, since x was arbitrary, we can claim that each element of $\bigcap_{j=1}^{m} A_j$ is an interior point. Therefore, $\bigcap_{j=1}^{m} A_j$ is open.

(or #3) Suppose that $A \subset S$ is closed and $x \in A^c$. Then $x \notin A$ and, because A contains all of its limit points, x is not a limit point of A. Hence, $x \notin A \wedge \neg (\forall N_r(x)) \left[A \cap (N_r(x) - \{x\}) \neq \emptyset\right]$ is true. It follows that $x \notin A$ and there exists a $\rho > 0$ such that $A \cap \left(N_\rho(x) - \{x\}\right) = \emptyset$. Thus, $A \cap N_\rho(x) = \emptyset$ and we conclude that $N_\rho(x) \subset A^c$; i.e., x is an interior point of A^c. Since x was arbitrary, we have that each element of A^c is an interior point. Therefore, A^c is open.

To prove the converse, suppose that $A \subset S$ is such that A^c is open. If p is a limit point of A, then $(\forall N_r(p)) \left[A \cap (N_r(p) - \{p\}) \neq \emptyset\right]$. But, for any $\rho > 0$, $A \cap \left(N_\rho(p) - \{p\}\right) \neq \emptyset$ implies that $\left(N_\rho(p) - \{p\}\right)$ is not contained in A^c. Hence,

p is not an interior point of A^c and we conclude that $p \notin A^c$. Therefore, $p \in A$. Since p was arbitrary, we have that A contains all of its limit points which yields that A is closed. ■

Remark 3.3.14 *Take the time to look back at the proof of (#2) to make sure that you where that fact that the intersection was over a finite family of open subsets of S was critical to the proof.*

Given a subset of a metric space that is neither open nor closed we'd like to have a way of describing the process of "extracting an open subset" or "building up to a closed subset." The following terminology will allow us to classify elements of a metric space S in terms of their relationship to a subset $A \subset S$.

Definition 3.3.15 *Let A be a subset of a metric space S. Then*

1. *A point $p \in S$ is an **exterior point** of A if and only if*

   $$(\exists N_r(p))\left[N_r(p) \subset A^c\right],$$

   where $A^c = S - A$.

2. *The **interior of** A, denoted by $\operatorname{Int}(A)$ or $A^{(0)}$, is the set of all interior points of A.*

3. *The **exterior of** A, denoted by $\operatorname{Ext}(A)$, is the set of all exterior points of A.*

4. *The **derived set** of A, denoted by A', is the set of all limit points of A.*

5. *The **closure of** A, denoted by $\overline{A}$, is the union of A and its derived set; i.e., $\overline{A} = A \cup A'$.*

6. *The **boundary of** A, denoted by ∂A, is the difference between the closure of A and the interior of A; i.e., $\partial A = \overline{A} - A^{(0)}$.*

Remark 3.3.16 *Note that, if A is a subset of a metric space S, then* $\operatorname{Ext}(A) = \operatorname{Int}(A^c)$ *and*

$$x \in \partial A \Leftrightarrow (\forall N_r(x))\left[N_r(x) \cap A \neq \emptyset \wedge N_r(x) \cap A^c \neq \emptyset\right].$$

The proof of these statements are left as exercises.

Excursion 3.3.17 *For $A \cup B$ where*

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

and

$$B = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - 1| \leq 1 \wedge |x_2 - 1| \leq 1\}.$$

1. *Sketch a graph of $A \cup B$.*

2. *On separate representations for $\mathbb{R}^2$, show each of the following*

$$\operatorname{Int}(A \cup B)\,,\ \operatorname{Ext}(A \cup B)\,,\ (A \cup B)'\,, \text{ and } \overline{(A \cup B)}.$$

***Hopefully, your graph of $A \cup B$ consisted of the union of the open disc that is centered at the origin and has radius one with the closed square having vertices

$(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$; the disc and square overlap in the first quadrant and the set is not open and not closed. Your sketch of $\text{Int}\,(A \cup B)$ should have shown the disc and square without the boundaries (;i.e., with the outline boundaries as not solid curves), while your sketch of $\text{Ext}\,(A \cup B)$ should have shown everything that is outside the combined disc and square–also with the outlining boundary as not solid curves. Finally, because $A \cup B$ has no isolated points, $(A \cup B)'$ and $\overline{(A \cup B)}$ are shown as the same sets–looking like $\text{Int}\,(A \cup B)$ with the outlining boundary now shown as solid curves.***

The following theorem relates the properties of being open or closed to the concepts described in Definition 3.3.15.

Theorem 3.3.18 *Let A be any subset of a metric space S.*

- *(a) The derived set of A, A', is a closed set.*
- *(b) The closure of A, $\overline{A}$, is a closed set.*
- *(c) Then $A = \overline{A}$ if and only if A is closed.*
- *(d) The boundary of A, ∂A, is a closed set.*
- *(e) The interior of A, $\text{Int}\,(A)$, is an open set.*
- *(f) If $A \subset B$ and B is closed, then $\overline{A} \subset B$.*
- *(g) If $B \subset A$ and B is open, $B \subset \text{Int}(A)$.*
- *(h) Any point (element) of S is a closed set.*

The proof of part *(a)* is problem #6 in WRp43, while (e) and (g) are parts of problem #9 in WRp43.

Excursion 3.3.19 *Fill in what is missing to complete the following proofs of parts (b), (c), and (f).*

Part (b): In view of Theorem 3.3.13(#3), it suffices to show that $\underset{(1)}{\underline{\hspace{3cm}}}$.

Suppose that $x \in S$ is such that $x \in \left(\overline{A}\right)^c$. Because $\overline{A} = A \cup A'$, it follows that $x \notin A$ and $\underset{(2)}{\underline{\hspace{3cm}}}$. *From the latter, there exists a neighborhood of x, $N\,(x)$, such* *that* $\left(\underset{(3)}{\underline{\hspace{3cm}}}\right) \cap A = \emptyset$*; while the former yields that* $\left(\underset{(4)}{\underline{\hspace{3cm}}}\right) \cap A = \emptyset$*. Hence,* $N(x) \subset A^c$*. Suppose that* $y \in N(x)$*. Since* $\underset{(5)}{\underline{\hspace{3cm}}}$*, there exists a neighborhood* $N^*(y)$ *such that* $N^*(y) \subset N(x)$*. From the transitivity of subset,* $\underset{(6)}{\underline{\hspace{3cm}}}$ *from which we conclude that* y *is not a limit point of* A*; i.e.,* $y \in (A')^c$*. Because* y *was arbitrary,*

$$(\forall y)\left[y \in N(x) \Rightarrow \underset{(7)}{\underline{\hspace{3cm}}}\right];$$

i.e., $\underset{(8)}{\underline{\hspace{6cm}}}$*. Combining our containments yields that* $N(x) \subset A^c$ *and* $\underset{(8)}{\underline{\hspace{6cm}}}$*. Hence,*

$$N(x) \subset A^c \cap (A')^c = \left[\underset{(9)}{\underline{\hspace{3cm}}}\right]^c.$$

Since x *was arbitrary, we have shown that*

$$\underset{(10)}{\underline{\hspace{9cm}}}.$$

Therefore, $\left(\overline{A}\right)^c$ *is open.*

Part (c): From part (b), if $A = \overline{A}$*, then* $\underset{(11)}{\underline{\hspace{3cm}}}$*.*

Conversely, if $\underset{(12)}{\underline{\hspace{3cm}}}$*, then* $A' \subset A$*. Hence,* $A \cup A' = \underset{(13)}{\underline{\hspace{1cm}}}$*; that is,* $\overline{A} = A$*.*

Part (f): Suppose that $A \subset B$*,* B *is closed, and* $x \in \overline{A}$*. Then* $x \in A$ *or* $\underset{(14)}{\underline{\hspace{3cm}}}$*. If* $x \in A$*, then* $x \in B$*; if* $x \in A'$*, then for every neighborhood of* x*,* $N(x)$*, there exists* $w \in A$ *such that* $w \neq x$ *and* $\underset{(15)}{\underline{\hspace{3cm}}}$*. But then*

$w \in B$ and $(N(x) - \{x\}) \cap B \neq \emptyset$. Since $N(x)$ was arbitrary, we conclude that $\underset{(16)}{\underline{\hspace{3cm}}}$. *Because B is closed,* $\underset{(17)}{\underline{\hspace{3cm}}}$. *Combining the conclusions and noting that $x \in \overline{A}$ was arbitrary, we have that*

$$(\forall x)\left[\underset{(18)}{\underline{\hspace{6cm}}}\right].$$

Thus, $\overline{A} \subset B$.

*******Acceptable responses are (1) the complement of A closure is open, (2) $x \notin A'$, (3) $N(x) - \{x\}$, (4) $N(x)$, (5) $N(x)$ is open, (6) $N^*(y) \subset A^c$, (7) $y \in (A')^c$, (8) $N(x) \subset (A')^c$; (9) $A \cup A'$, and (10) $(\forall x)\left(x \in \left(\overline{A}\right)^c \Rightarrow (\exists N_r(x))\left(N_r(x) \subset \left(\overline{A}\right)^c\right)\right)$; (11) A is closed, (12) A is closed, (13) A; (14) x is a limit point of A (or $x \in A'$); (15) $w \in N(x)$; (16) x is a limit point of B (or $x \in B'$); (17) $x \in B$, (18) $x \in \overline{A} \Rightarrow x \in B$.*******

Definition 3.3.20 *For a metric space (X, d) and $E \subset X$, the set E is* ***dense*** *in X if and only if*

$$(\forall x)\left(x \in X \Rightarrow x \in E \vee x \in E'\right).$$

Remark 3.3.21 *Note that for a metric space (X, d), $E \subset X$ implies that $\overline{E} \subset X$ because the space X is closed. On the other hand, if E is dense in X, then $X \subset E \cup E' = \overline{E}$. Consequently, we see that E is dense in a metric space X if and only if $\overline{E} = X$.*

Example 3.3.22 *We have that the sets of rationals and irrationals are dense in Euclidean* 1*-space. This was shown in the two Corollaries the Archimedean Principle for Real Numbers that were appropriately named "Density of the Rational Numbers" and "Density of the Irrational Numbers."*

Definition 3.3.23 *For a metric space (X, d) and $E \subset X$, the set E is* ***bounded*** *if and only if*

$$(\exists M)(\exists q)\left[M \in \mathbb{R}^+ \wedge q \in X \wedge (E \subset N_M(q))\right].$$

Excursion 3.3.24 *Justify that each of the following sets is bounded in Euclidean space.*

1. $A = \left\{(x_1, x_2) \in \mathbb{R}^2 : -1 \le x_1 < 2 \wedge |x_2 - 3| < 1\right\}$

2. $B = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \ge 0 \wedge x_2 \ge 0 \wedge x_3 \ge 0 \wedge 2x_1 + x_2 + 4x_3 = 2\right\}$

Remark 3.3.25 *Note that, for* $(\mathbb{R}^2, \hat{d})$*, where*

$$\hat{d}(x, y) = \begin{cases} 0, & \textit{if} \quad x = y \\ \\ 1, & \textit{if} \quad x \neq y \end{cases},$$

the space $\mathbb{R}^2$ *is bounded. This example stresses that classification of a set as bounded is tied to the metric involved and may allow for a set to be bounded*

The definitions of least upper bound and greatest lower bound directly lead to the observation that they are limit points for bounded sets of real numbers.

Theorem 3.3.26 *Let* E *be a nonempty set of real numbers that is bounded,* $\alpha = \sup(E)$*, and* $\beta = \inf(E)$*. Then* $\alpha \in \overline{E}$ *and* $\beta \in \overline{E}$*.*

Space for illustration.

Proof. It suffices to show the result for least upper bounds. Let E be a nonempty set of real numbers that is bounded above and $\alpha = \sup(E)$. If $\alpha \in E$, then $\alpha \in \overline{E} =$

$E \cup E'$. For $\alpha \notin E$, suppose that h is a positive real number. Because $\alpha - h < \alpha$ and $\alpha = \sup(E)$, there exists $x \in E$ such that $\alpha - h < x < \alpha$. Since h was arbitrary,

$$(\forall h)\,(h > 0 \Rightarrow (\exists x)\,(\alpha - h < x < \alpha))\,;$$

i.e., α is a limit point for E. Therefore, $\alpha \in \overline{E}$ as needed. ■

Remark 3.3.27 *In view of the theorem we note that any closed nonempty set of real numbers that is bounded above contains its least upper bound and any closed nonempty set of real numbers that is bounded below contains its greatest lower bound.*

3.3.2 Open Relative to Subsets of Metric Spaces

Given a metric space (X, d), for any subset Y of X, $d \restriction_Y$ is a metric on Y. For example, given the Euclidean metric d_e on $\mathbb{R}^2$ we have that $d_e \restriction_{\mathbb{R}\times\{0\}}$ corresponds to the (absolute value) Euclidean metric, $d = |x - y|$, on the reals. It is natural to ask about how properties studied in the (parent) metric space transfer to the subset.

Definition 3.3.28 *Given a metric space (X, d) and $Y \subset X$. A subset E of Y is **open relative to** Y if and only if*

$$(\forall p)\left[p \in E \Rightarrow (\exists r)\left(r > 0 \wedge (\forall q)\left[q \in Y \wedge d(p,q) < r \Rightarrow q \in E\right]\right)\right]$$

which is equivalent to

$$(\forall p)\left[p \in E \Rightarrow (\exists r)\,(r > 0 \wedge Y \cap N_r(p) \subset E)\right].$$

Example 3.3.29 *For Euclidean 2-space, $\left(\mathbb{R}^2, d\right)$, consider the subsets*

$$Y = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 3\right\} \text{ and } Z = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \wedge 2 \leq x_2 < 5\right\}.$$

(a) The set $X_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 : 3 \leq x_1 < 5 \wedge 1 < x_2 < 4\right\} \cup \{(3,1), (3,4)\}$ is not open relative to Y, while $X_2 = \left\{(x_1, x_2) \in \mathbb{R}^2 : 3 \leq x_1 < 5 \wedge 1 < x_2 < 4\right\}$ is open relative to Y.

(b) The half open interval $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \wedge 2 \leq x_2 < 3\right\}$ is open relative to Z.

From the example we see that a subset of a metric space can be open relative to another subset though it is not open in the whole metric space. On the other hand, the following theorem gives us a characterization of open relative to subsets of a metric space in terms of sets that are open in the metric space.

Theorem 3.3.30 *Suppose that* (X, d) *is a metric space and* $Y \subset X$*. A subset* E *of* Y *is open relative to* Y *if and only if there exists an open subset* G *of* X *such that* $E = Y \cap G$*.*

Space for scratch work.

Proof. Suppose that (X, d) is a metric space, $Y \subset X$, and $E \subset Y$.

If E is open relative to Y, then corresponding to each $p \in E$ there exists a neighborhood of p, $N_{r_p}(p)$, such that $Y \cap N_{r_p}(p) \subset E$. Let $\mathcal{A} = \left\{N_{r_p}(p) : p \in E\right\}$. By Theorems 3.3.5(a) and 3.3.13(#1), $G \underset{def}{=} \cup\mathcal{A}$ is an open subset of X. Since $p \in N_{r_p}(p)$ for each $p \in E$, we have that $E \subset G$ which, with $E \subset Y$, implies that $E \subset G \cap Y$. On the other hand, the neighborhoods $N_{r_p}(p)$ were chosen such that $Y \cap N_{r_p}(p) \subset E$; hence,

$$\bigcup_{p\in E}\left(Y \cap N_{r_p}(p)\right) = Y \cap \left(\bigcup_{p\in E} N_{r_p}(p)\right) = Y \cap G \subset E.$$

Therefore, $E = Y \cap G$, as needed.

Now, suppose that G is an open subset of X such that $E = Y \cap G$ and $p \in E$. Then $p \in G$ and G open in X yields the existence of a neighborhood of p, $N(p)$, such that $N(p) \subset G$. It follows that $N(p) \cap Y \subset G \cap Y = E$. Since p was arbitrary, we have that

$$(\forall p)\left[p \in E \Rightarrow (\exists N(p))\left[N(p) \cap Y \subset E\right]\right];$$

i.e., E is open relative to Y. ■

3.3.3 Compact Sets

In metric spaces, many of the properties that we study are described in terms of neighborhoods. The next set characteristic will allow us to extract finite collections of neighborhoods which can lead to bounds that are useful in proving other results about subsets of metric spaces or functions on metric spaces.

Definition 3.3.31 *Given a metric space* (X, d) *and* $A \subset X$*, the family* $\{G_\alpha : \alpha \in \Delta\}$ *of subsets of* X *is an* ***open cover*** *for* A *if and only if* G_α *is open for each* $\alpha \in \Delta$ *and* $A \subset \bigcup_{\alpha \in \Delta} G_\alpha$.

Definition 3.3.32 *A subset* K *of a metric space* (X, d) *is* ***compact*** *if and only if every open cover of* K *has a finite subcover; i.e., given any open cover* $\{G_\alpha : \alpha \in \Delta\}$ *of* K*, there exists an* $n \in \mathbb{J}$ *such that* $\left\{G_{\alpha_k} : k \in \mathbb{J} \wedge 1 \leq k \leq n\right\}$ *is a cover for* K.

We have just seen that a subset of a metric space can be open relative to another subset without being open in the whole metric space. Our first result on compact sets is tells us that the situation is different when we look at compactness relative to subsets.

Theorem 3.3.33 *For a metric space* (X, d)*, suppose that* $K \subset Y \subset X$*. Then* K *is compact relative to* X *if and only if* K *is compact relative to* Y.

Excursion 3.3.34 *Fill in what is missing to complete the following proof of Theorem 3.3.33.*

Space for scratch work.

Proof. *Let* (X, d) *be a metric space and* $K \subset Y \subset X$.

Suppose that K *is compact relative to* X *and* $\{U_\alpha : \alpha \in \Delta\}$ *is a family of sets such that, for each* α, U_α *is open relative to* Y *such that*

$$K \subset \bigcup_{\alpha\in\Delta} U_\alpha .$$

By Theorem 3.3.30, corresponding to each $\alpha \in \Delta$*, there exists a set* G_α *such that* G_α *is open relative to* X *and* $\underset{(1)}{\underline{\hspace{3cm}}}$.

Since $K \subset Y$ *and*

$K \subset \bigcup_{\alpha\in\Delta} U_\alpha = \bigcup_{\alpha\in\Delta}\left(\underset{(1)}{\underline{\hspace{3cm}}}\right) = Y \cap \bigcup_{\alpha\in\Delta} G_\alpha$*, if follows that*

$$K \subset \bigcup_{\alpha\in\Delta} G_\alpha .$$

Because K *is compact relative to* X*, there exists a finite number of elements of* Δ*,* $\alpha_1, \alpha_2, ..., \alpha_n$*, such that*

$$\underset{(2)}{\underline{\hspace{6cm}}}.$$

Now $K \subset Y$ *and* $K \subset \bigcup_{j=1}^{n} G_{\alpha_j}$ *yields that*

$$K \subset Y \cap \bigcup_{j=1}^{n} G_{\alpha_j} = \underset{(3)}{\underline{\hspace{3cm}}} = \underset{(4)}{\underline{\hspace{3cm}}}.$$

Since $\{U_\alpha : \alpha \in \Delta\}$ *was arbitrary, we have shown that every open relative to* Y *cover of* K *has a finite subcover. Therefore,* $\underset{(5)}{\underline{\hspace{6cm}}}$.

Conversely, suppose that K *is compact relative to* Y *and that* $\{W_\alpha : \alpha \in \Delta\}$ *is a family of sets such that, for each* α, W_α *is open relative to* X *and*

$$K \subset \bigcup_{\alpha\in\Delta} W_\alpha .$$

For each $\alpha \in \Delta$, *let* $U_\alpha = Y \cap W_\alpha$. *Now* $K \subset Y$ *and* $K \subset \bigcup_{\alpha\in\Delta} W_\alpha$ *implies that*

$$\underset{(6)}{\underline{\hspace{6cm}}}.$$

Consequently, $\{U_\alpha : \alpha \in \Delta\}$ *is an open relative to* Y *cover for* K. *Now* K *compact relative to* Y *yields that there exists a finite number of elements of* Δ, $\alpha_1, \alpha_2, ..., \alpha_n$, *such that* $\underset{(7)}{\underline{\hspace{3cm}}}$. *Since*

$$\bigcup_{j=1}^{n} U_{\alpha_j} = \bigcup_{j=1}^{n} \left(Y \cap W_{\alpha_j}\right) = Y \cap \bigcup_{j=1}^{n} W_{\alpha_j}$$

and $K \subset Y$, *it follows that* $\underset{(8)}{\underline{\hspace{4cm}}}$.
Since $\{W_\alpha : \alpha \in \Delta\}$ *was arbitrary, we conclude that every family of sets that form an open relative to* X *cover of* K *has a finite subcover. Therefore,*

$\underset{(9)}{\underline{\hspace{6cm}}}$.

■

Acceptable fill-ins: (1) $U_\alpha = Y \cap G_\alpha$, (2) $K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$ (or $K \subset \bigcup_{j=1}^{n} G_{\alpha_j}$), (3) $\bigcup_{j=1}^{n} \left(Y \cap G_{\alpha_j}\right)$, (4) $\bigcup_{j=1}^{n} U_{\alpha_j}$, (5) K is compact relative to Y, (6) $K \subset Y \cap \bigcup_{\alpha\in\Delta} W_\alpha = \bigcup_{\alpha\in\Delta} (Y \cap W_\alpha) = \bigcup_{\alpha\in\Delta} U_\alpha$, (7) $K \subset \bigcup_{j=1}^{n} U_{\alpha_j}$, (8) $K \subset \bigcup_{j=1}^{n} W_{\alpha_j}$, (9) K is compact in X.

Our next set of results show relationships between the property of being compact and the property of being closed.

Theorem 3.3.35 *If* A *is a compact subset of a metric space* (S, d), *then* A *is closed.*

Excursion 3.3.36 *Fill-in the steps of the proof as described*

Proof. *Suppose that* A *is a compact subset of a metric space* (S, d) *and* $p \in S$ *is such that* $p \notin A$. *For* $q \in A$, *let* $r_q = \frac{1}{4} d(p, q)$. *The* $\left\{N_{r_q}(q) : q \in A\right\}$ *is an open cover for* A. *Since* A *is compact, there exists a finite number of* q, *say* $q_1, q_2, ..., q_n$, *such that*

$$A \subset N_{r_{q_1}}(q_1) \cup N_{r_{q_2}}(q_2) \cup \cdots \cup N_{r_{q_n}}(q_n) \underset{def}{=} W.$$

(a) Justify that the set $V = N_{r_{q_1}}(p) \cap N_{r_{q_2}}(p) \cap \cdots \cap N_{r_{q_n}}(p)$ *is a neighborhood of* p *such that* $V \cap W = \emptyset$.

(b) Justify that A^c *is open.*

(c) Justify that the result claimed in the theorem is true.

■

For (a), hopefully you noted that taking $r = \min_{1 \leq j \leq n} r_{q_j}$ yields that $N_{r_{q_1}}(p) \cap N_{r_{q_2}}(p) \cap \cdots \cap N_{r_{q_n}}(p) = N_r(p)$. To complete (b), you needed to observe that $N_r(p) \subset A^c$ made p an interior point of A^c; since p was an arbitrary point satisfying $p \notin A$, it followed that A^c is open. Finally, part (c) followed from Theorem 3.3.13(#3) which asserts that the complement of an open set is closed; thus, $(A^c)^c = A$ is closed.

Theorem 3.3.37 *In any metric space, closed subsets of a compact sets are compact.*

Space for scratch work.

Excursion 3.3.38 *Fill in the two blanks in order to complete the following proof of the theorem.*

Proof. For a metric space (X, d), suppose that $F \subset K \subset X$ are such that F is closed (relative to X) and K is compact. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ be an open cover for F. Then the family $\Omega = \{V : V \in \mathcal{G} \vee V = F^c\}$ is an open cover for K. It follows from K being compact that there exists a finite number of elements of Ω, say $V_1, V_2, \ldots, V_n$, such that

________________________.

Because $F \subset K$, we also have that

________________________.

If for some $j \in \mathbb{J}$, $1 \leq j \leq n$, $F^c = V_j$, the family $\{V_k : 1 \leq k \leq n \wedge k \neq j\}$ would still be a finite open cover for F. Since $\mathcal{G}$ was an arbitrary open cover for F, we conclude that every open cover of F has a finite subcover. Therefore, F is compact. ■

Corollary 3.3.39 *If F and K are subsets of a metric space such that F is closed and K is compact, then $F \cap K$ is compact.*

Proof. As a compact subset of a metric space, from Theorem 3.3.35, K is closed. Then, it follows directly from Theorems 3.3.13(#5) and 3.3.37 that $F \cap K$ is compact as a closed subset of the compact set K. ■

Remark 3.3.40 *Notice that Theorem 3.3.35 and Theorem 3.3.37 are not converses of each other. The set $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 2 \wedge x_2 = 0\right\}$ is an example of a closed set in Euclidean 2-space that is not compact.*

Definition 3.3.41 *Let* $\{S_n\}_{n=1}^{\infty}$ *be a sequence of subsets of a metric space* X. *Then* $\{S_n\}_{n=1}^{\infty}$ *is a* ***nested sequence of sets*** *if and only if* $(\forall n)\ (n \in \mathbb{J} \Rightarrow S_{n+1} \subset S_n)$.

Definition 3.3.42 *A family* $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ *of sets in the universe* $\mathcal{U}$ *has the* ***finite intersection property*** *if and only if the intersection over any finite subfamily of* $\mathcal{A}$ *is nonempty; i.e.,*

$$(\forall \Omega)\left[\Omega \subset \Delta \wedge \Omega \text{ finite} \Rightarrow \bigcap_{\beta \in \Omega} A_\beta \neq \emptyset\right].$$

The following theorem gives a sufficient condition for a family of nonempty compact sets to be disjoint. The condition is not being offered as something for you to apply to specific situations; it leads us to a useful observation concerning nested sequences of nonempty compact sets.

Theorem 3.3.43 *If* $\{K_\alpha : \alpha \in \Delta\}$ *is a family of nonempty compact subsets of a metric space* X *that satisfies the finite intersection property, then* $\bigcap_{\alpha \in \Delta} K_\alpha \neq \emptyset$.

Space for notes.

Proof. Suppose that $\bigcap_{\alpha \in \Delta} K_\alpha = \emptyset$ and choose $K_\delta \in \{K_\alpha : \alpha \in \Delta\}$. Since $\bigcap_{\alpha \in \Delta} K_\alpha = \emptyset$,

$$(\forall x)\left[x \in K_\delta \Rightarrow x \notin \bigcap_{\alpha \in \Delta} K_\alpha\right].$$

Let

$$\mathcal{G} = \{K_\alpha : \alpha \in \Delta \wedge K_\alpha \neq K_\delta\}.$$

Because each K_α is compact, by Theorems 3.3.35 and 3.3.13(#3), K_α is closed and K_α^c is open. For any $w \in K_\delta$, we have that $w \notin \bigcap_{\alpha \in \Delta} K_\alpha$. Hence, there exists a

$\beta \in \Delta$ such that $w \notin K_\beta$ from which we conclude that $w \in K_\beta^c$ and $K_\beta \neq K_\delta$. Since w was arbitrary, we have that

$$(\forall w)\left[w \in K_\delta \Rightarrow (\exists \beta)\left(\beta \in \Delta \wedge K_\beta \neq K_\delta \wedge w \in K_\beta^c\right)\right].$$

Thus, $K_\delta \subset \bigcup_{G \in \mathcal{G}} G$ which establishes $\mathcal{G}$ as an open cover for K_δ. Because K_δ is compact there exists a finite number of elements of $\mathcal{G}$, $K_{\alpha_1}^c$, $K_{\alpha_2}^c$, ..., $K_{\alpha_n}^c$, such that

$$K_\delta \subset \bigcup_{j=1}^{n} K_{\alpha_j}^c = \left(\bigcap_{j=1}^{n} K_{\alpha_j}\right)^c$$

from DeMorgan's Laws from which it follows that

$$K_\delta \cap \left(\bigcap_{j=1}^{n} K_{\alpha_j}\right) = \emptyset.$$

Therefore, there exists a finite subfamily of $\{K_\alpha\}$ that is disjoint.

We have shown that if $\bigcap_{\alpha \in \Delta} K_\alpha = \emptyset$, then there exists a finite subfamily of $\{K_\alpha : \alpha \in \Delta\}$ that has empty intersection. From the Contrapositive Tautology, if $\{K_\alpha : \alpha \in \Delta\}$ is a family of nonempty compact subsets of a metric space such that the intersection of any finite subfamily is nonempty, then $\bigcap_{\alpha \in \Delta} K_\alpha \neq \emptyset$. ■

Corollary 3.3.44 *If $\{K_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty compact sets, then $\bigcap_{n \in \mathbb{J}} K_n \neq \emptyset$.*

Proof. For Δ any finite subset of $\mathbb{J}$, let $m = \max\{j : j \in \Delta\}$. Because $\{K_n\}_{n=1}^{\infty}$ is a nested sequence on nonempty sets, $K_m \subset \bigcap_{j \in \Delta} K_j$ and $\bigcap_{j \in \Delta} K_j \neq \emptyset$. Since Δ was arbitrary, we conclude that $\{K_n : n \in \mathbb{J}\}$ satisfies the finite intersection property. Hence, by Theorem 3.3.43, $\bigcap_{n \in \mathbb{J}} K_n \neq \emptyset$. ■

Corollary 3.3.45 *If $\{S_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty closed subsets of a compact sets in a metric space, then $\bigcap_{n \in \mathbb{J}} S_n \neq \emptyset$.*

Theorem 3.3.46 *In a metric space, any infinite subset of a compact set has a limit point in the compact set.*

Space for notes and/or scratch work.

Proof. Let K be a compact subset of a metric space and E is a nonempty subset of K. Suppose that no element of K is a limit point for E. Then for each x in K there exists a neighborhood of x, say $N(x)$, such that $(N(x) - \{x\}) \cap E = \emptyset$. Hence, $N(x)$ contains at most one point from E; namely x. The family $\{N(x) : x \in K\}$ forms an open cover for K. Since K is compact, there exists a finite number of elements in $\{N(x) : x \in K\}$, say $N(x_1), N(x_2), ..., N(x_n)$, such that $K \subset N(x_1) \cup N(x_2) \cup \cdots \cup N(x_n)$. Because $E \subset K$, we also have that $E \subset N(x_1) \cup N(x_2) \cup \cdots \cup N(x_n)$. From the way that the neighborhoods were chosen, it follows that $E \subset \{x_1, x_2, ..., x_n\}$. Hence, E is finite.

We have shown that for any compact subset K of metric space, every subset of K that has not limit points in K is finite. Consequently, any infinite subset of K must have at least one limit point that is in K. ■

3.3.4 Compactness in Euclidean n-space

Thus far our results related to compact subsets of metric spaces described implications of that property. It would be nice to have some characterizations for compactness. In order to achieve that goal, we need to restrict our consideration to specific metric spaces. In this section, we consider only real n-space with the Euclidean metric. Our first goal is to show that every n-cell is compact in $\mathbb{R}^n$. Leading up to this we will show that every nested sequence of nonempty n-cells is not disjoint.

Theorem 3.3.47 (Nested Intervals Theorem) *If $\{I_n\}_{n=1}^{\infty}$ is a nested sequence of intervals in $\mathbb{R}^1$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.*

Proof. For the nested sequence of intervals $\{I_n\}_{n=1}^{\infty}$, let $I_n = [a_n, b_n]$ and $A = \{a_n : n \in \mathbb{J}\}$. Because $\{I_n\}_{n=1}^{\infty}$ is nested, $[a_n, b_n] \subset [a_1, b_1]$ for each $n \in \mathbb{J}$. It follows that $(\forall n)\ (n \in \mathbb{J} \Rightarrow a_n \leq b_1)$. Hence, A is a nonempty set of real numbers that is bounded above. By the Least Upper Bound Property, $x \underset{def}{=} \sup A$ exists and is real. From the definition of least upper bound, $a_n \leq x$ for each $n \in \mathbb{J}$. For any positive integers k and m, we have that

$$a_k \leq a_{k+m} \leq b_{k+m} \leq b_k$$

from which it follows that $x \leq b_n$ for all $n \in \mathbb{J}$. Since $a_n \leq x \leq b_n$ for each $n \in J$, we conclude that $x \in \bigcap_{n=1}^{\infty} I_n$. Hence, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. ■

Remark 3.3.48 *Note that, for* $B = \{b_n : n \in J\}$ *appropriate adjustments in the proof that was given for the Nested Intervals Theorem would allow us to conclude that* $\inf B \in \bigcap_{n=1}^{\infty} I_n$. *Hence, if lengths of the nested integrals go to* 0 *as n goes to* ∞, *then* $\sup A = \inf B$ *and we conclude that* $\bigcap_{n=1}^{\infty} I_n$ *consists of one real number.*

The Nested Intervals Theorem generalizes to nested n-cells. The key is to have the set-up that makes use of the n intervals $\left[x_j, y_j\right]$, $1 \leq j \leq n$, that can be associated with $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$.

Theorem 3.3.49 (Nested n-Cells Theorem) *Let n be a positive integer. If* $\{I_k\}_{k=1}^{\infty}$ *is a nested sequence of n-cells, then* $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$.

Proof. For the nested sequence of intervals $\{I_k\}_{k=1}^{\infty}$, let

$$I_k = \left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : a_{k,j} \leq x_j \leq b_{k,j} \text{ for } j = 1, 2, ..., n\right\}.$$

For each j, $1 \leq j \leq n$, let $I_{k,j} = \left[a_{k,j}, b_{k,j}\right]$. Then each $\left\{I_{k,j}\right\}_{k=1}^{\infty}$ satisfies the conditions of the Nested Intervals Theorem. Hence, for each j, $1 \leq j \leq n$, there exists $w_j \in R$ such that $w_j \in \bigcap_{k=1}^{\infty} I_{k,j}$. Consequently, $(w_1, w_2, ..., w_n) \in \bigcap_{k=1}^{\infty} I_k$ as needed. ■

Theorem 3.3.50 *Every n-cell is compact.*

Proof. For real constants $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ such that $a_j < b_j$ for each $j = 1, 2, ..., n$, let

$$I_0 = I = \left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : (\forall j \in \mathbb{J})\left(1 \le j \le n \Rightarrow a_j \le x_j \le b_j\right)\right\}$$

and

$$\delta = \sqrt{\sum_{j=1}^{n} \left(b_j - a_j\right)^2}.$$

Then $(\forall \mathbf{x})(\forall \mathbf{y})\left[\mathbf{x}, \mathbf{y} \in I_0 \Rightarrow |\mathbf{x} - \mathbf{y}| \le \delta\right]$. Suppose that I_0 is not compact. Then there exists an open cover $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of I_0 for which no finite subcollection covers I_0. Now we will describe the construction of a nested sequence of n-cells each member of which is not compact. Use the space provided to sketch appropriate pictures for $n = 1$, $n = 2$, and $n = 3$ that illustrate the described construction.

For each j, $1 \le j \le n$, let $c_j = \dfrac{a_j + b_j}{2}$. The sets of intervals

$$\left\{\left(a_j, c_j\right) : 1 \le j \le n\right\} \qquad \textit{and} \qquad \left\{\left(c_j, b_j\right) : 1 \le j \le n\right\}$$

can be used to determine or generate 2^n new n-cells, $I_k^{(1)}$ for $1 \le k \le 2^n$. For example, each of

$$\left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : (j \in \mathbb{J})\left(1 \le j \le n \Rightarrow a_j \le x_j \le c_j\right)\right\},$$

$$\left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : (\forall j \in \mathbb{J})\left(1 \le j \le n \Rightarrow c_j \le x_j \le b_j\right)\right\},$$

and

$$\left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : a_j \leq x_j \leq c_j \text{ if } 2 \mid j \text{ and } c_j \leq x_j \leq b_j \text{ if } 2 \nmid j\right\}$$

is an element of $\left\{I_k^{(1)} : 1 \leq k \leq 2^n\right\}$. For each $k \in \mathbb{J}$, $1 \leq k \leq 2^n$, $I_k^{(1)}$ is a subset (sub-n-cell) of I_0 and $\bigcup_{k=1}^{2^n} I_k^{(1)} = I_0$. Consequently, $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ is an open cover for each of the 2^n sub-n-cells. Because I_0 is such that no finite subcollection from $\mathcal{G}$ covers I_0, it follows that at least one of the elements of $\left\{I_k^{(1)} : 1 \leq k \leq 2^n\right\}$ must also satisfy that property. Let I_1 denote an element of $\left\{I_k^{(1)} : 1 \leq k \leq 2^n\right\}$ for which no finite subcollection from $\mathcal{G}$ covers I_1. For $(x_1, x_2, ..., x_n) \in I_1$ we have that either $a_j \leq x_j \leq c_j$ or $c_j \leq x_j \leq b_j$ for each j, $1 \leq j \leq n$. Since

$$\frac{c_j - a_j}{2} = \frac{b_j - c_j}{2} = \frac{b_j - a_j}{2},$$

it follows that, for $\mathbf{x} = (x_1, x_2, ..., x_n)$, $\mathbf{y} = (y_1, y_2, ..., y_n) \in I_1$

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{j=1}^{n} \left(y_j - x_j\right)^2} \leq \sqrt{\sum_{j=1}^{n} \frac{\left(b_j - a_j\right)^2}{2^2}} = \frac{\delta}{2};$$

i.e., the diam (I_1) is $\dfrac{\delta}{2}$.

The process just applied to I_0 to obtain I_1 can not be applied to obtain a sub-n-cell of I_1 that has the transferred properties. That is, if

$$I_1 = \left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : (\forall j \in \mathbb{J}) \left(1 \leq j \leq n \Rightarrow a_j^{(1} \leq x_j \leq b_j^{(1)}\right)\right\},$$

letting $c_j^{(1)} = \dfrac{a_j^{(1)} + b_j^{(1)}}{2}$ generates two set of intervals

$$\left\{\left(a_j^{(1)}, c_j^{(1)}\right) : 1 \leq j \leq n\right\} \qquad and \qquad \left\{\left(c_j^{(1)}, b_j^{(1)}\right) : 1 \leq j \leq n\right\}$$

that will determine 2^n new n-cells, $I_k^{(2)}$ for $1 \leq k \leq 2^n$, that are sub-n-cells of I_1. Now, since $\mathcal{G}$ is an open cover for I_1 such that no finite subcollection from $\mathcal{G}$ covers I_1 and $\bigcup_{k=1}^{2^n} I_k^{(2)} = I_1$, it follows that there is at least one element of $\left\{I_k^{(2)} : 1 \leq k \leq 2^n\right\}$ that cannot be covered with a finite subcollection from $\mathcal{G}$; choose one of those elements and denote it by I_2. Now the choice of $c_j^{(1)}$ allows us to show that $\operatorname{diam}(I_2) = \dfrac{\operatorname{diam}(I_1)}{2} = \dfrac{\delta}{2^2}$. Continuing this process generates $\{I_k\}_{k=0}^{\infty}$ that satisfies each of the following properties:

- $\{I_k\}_{k=0}^{\infty}$ is a nested sequence of n-cells,
- for each $k \in \mathbb{J}$, no finite subfamily of $\mathcal{G}$ covers I_k, and
- $(\forall \mathbf{x})\,(\forall \mathbf{y})\left[\mathbf{x}, \mathbf{y} \in I_k \Rightarrow |\mathbf{x} - \mathbf{y}| \leq 2^{-k}\delta\right]$.

From the Nested n-cells Theorem, $\bigcap\limits_{k=0}^{\infty} I_k \neq \emptyset$. Let $\zeta \in \bigcap\limits_{k=0}^{\infty} I_k$. Because $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ is an open cover for I_0 and $\bigcap\limits_{k=0}^{\infty} I_k \subset I_0$, there exists $G \in \mathcal{G}$ such that $\zeta \in G$. Since G is open, we there is a positive real number r such that $N_r(\zeta) \subset G$. Now $\operatorname{diam}(N_r(\zeta)) = 2r$ and, for $n \in \mathbb{J}$ large enough, $\operatorname{diam}(I_n) = 2^{-n}\delta < 2r$. Now, $\zeta \in I_k$ for all $k \in \mathbb{J}$ assures that $\zeta \in I_k$ for all $k \geq n$. Hence, for all $k \in \mathbb{J}$ such that $k \geq n$, $I_k \subset N_r(\zeta) \subset G$. In particular, each I_k, $k \geq n$, can be covered by one element of $\mathcal{G}$ which contradicts the method of choice that is assured if I_0 is not compact. Therefore, I_0 is compact. ■

The next result is a classical result in analysis. It gives us a characterization for compactness in real n-space that is simple; most of the "hard work" for the proof was done in when we proved Theorem 3.3.50.

Theorem 3.3.51 (The Heine-Borel Theorem) *Let A be a subset of Euclidean n-space. Then A is compact if and only if A is closed and bounded.*

Proof. Let A be a subset of Euclidean n-space $(\mathbb{R}^n, d)$

Suppose that A is closed and bounded. Then there exists an n-cell I such that $A \subset I$. For example, because A is bounded, there exists $M > 0$ such that $A \subset N_M\left(\overrightarrow{\mathbf{0}}\right)$; for this case, the n-cell

$$I = \left\{(x_1, x_2, ..., x_n) \in R^n : \max_{1 \leq j \leq n} \left|x_j\right| \leq M + 1\right\}$$

satisfies the specified condition. From Theorem 3.3.50, I is compact. Since $A \subset I$ and A is closed, it follows from Theorem 3.3.37 that A is compact.

Suppose that A is a compact subset of Euclidean n-space. From Theorem 3.3.35, we know that A is closed. Assume that A is not bounded and let $p_1 \in A$. Corresponding to each $m \in \mathbb{J}$, choose a p_m in A such that $p_m \neq p_k$ for $k = 1, 2, ..., (m-1)$ and $d(p_1, p_m) > m - 1$. As an infinite subset of the compact set A, by Theorem 3.3.46, $\{p_m : m \in \mathbb{J}\}$ has a limit point in A. Let $q \in A$ be a limit point for $\{p_m : m \in \mathbb{J}\}$. Then, for each $t \in \mathbb{J}$, there exists $p_{m_t} \in \{p_m : m \in \mathbb{J}\}$ such that $d\left(p_{m_t}, q\right) < \dfrac{1}{t+1}$. From the triangular inequality, it follows that for any $p_{m_t} \in \{p_m : m \in \mathbb{J}\}$,

$$d\left(p_{m_t}, p_1\right) \leq d\left(p_{m_t}, q\right) + d(q, p_1) < \frac{1}{1+t} + d(q, p_1) < 1 + d(q, p_1).$$

But $1+d(q, p_1)$ is a fixed real number, while p_{m_t} was chosen such that $d\left(p_{m_t}, p_1\right) > m_t - 1$ and $m_t - 1$ goes to infinity as t goes to infinity. Thus, we have reached a contradiction. Therefore, A is bounded. ■

The next theorem gives us another characterization for compactness. It can be shown to be valid over arbitrary metric spaces, but we will show it only over real n-space.

Theorem 3.3.52 *Let A be a subset of Euclidean n-space. Then A is compact if and only if every infinite subset of A has a limit point in A.*

Excursion 3.3.53 *Fill in what is missing in order to complete the following proof of Theorem 3.3.52.*

Proof. If A is a compact subset of Euclidean n-space, then every infinite subset of A has a limit point in A by Theorem 3.3.46.

Suppose that A is a subset of Euclidean n-space for which every infinite subset of A has a limit point in A. We will show that this assumption implies that A is closed and bounded. Suppose that w is a limit point of A. Then, for each $n \in \mathbb{J}$, there exists an x_n such that

$$x_n \in N_{\frac{1}{n}}(w) - \{w\}.$$

Let $S = \{x_n : n \in \mathbb{J}\}$. Then S is an $\underset{(1)}{\underline{\hspace{6cm}}}$ of A. Consequently, S has $\underset{(2)}{\underline{\hspace{6cm}}}$ in A. But S has only one limit point;

namely $\underset{(3)}{\underline{\hspace{1.5cm}}}$. Thus, $w \in A$. Since w was arbitrary, we conclude that A contains all of its limit point; i.e., $\underset{(4)}{\underline{\hspace{6cm}}}$.

Suppose that A is not bounded. Then, for each $n \in \mathbb{J}$, there exists y_n such that $|y_n| > n$. Let $S = \{y_n : n \in \mathbb{J}\}$. Then S is an $\underset{(5)}{\underline{\hspace{6cm}}}$ of A that has no finite limit point in A. Therefore,

$$A \text{ not bounded} \Rightarrow (\exists S)\left(S \subset A \wedge S \text{ is infinite } \wedge S \cap A' = \emptyset\right);$$

taking the contrapositive and noting that $\neg(P \wedge Q \wedge M)$ is logically equivalent to $[(P \wedge Q) \Rightarrow M]$ for any propositions P, Q and M, we conclude that

$$(\forall S)\left[\left(\underset{(6)}{\underline{\hspace{6cm}}}\right) \Rightarrow S \cap A' \neq \emptyset\right] \Rightarrow \underset{(7)}{\underline{\hspace{6cm}}}.$$

■

Acceptable completions include: (1) infinite subset, (2) a limit point, (3) w, (4) A is closed, (5) infinite subset, (6) $S \subset A \wedge S$ is infinite, and (7) A is bounded.

As an immediate consequence of Theorems 3.3.50 and 3.3.46, we have the following result that is somewhat of a generalization of the Least Upper Bound Property to n-space.

Theorem 3.3.54 (Weierstrass) *Every bounded infinite subset of Euclidean n-space has a limit point in $\mathbb{R}^n$.*

3.3.5 Connected Sets

With this section we take a brief look at one mathematical description for a subset of a metric space to be “in one piece.” This is one of those situations where “we recognize it when we see it,” at least with simply described sets in $\mathbb{R}$ and $\mathbb{R}^2$. The concept is more complicated than it seems since it needs to apply to all metric spaces and, of course, the mathematical description needs to be precise. Connectedness is defined in terms of the absence of a related property.

Definition 3.3.55 *Two subsets A and B of a metric space X are* ***separated*** *if and only if*

$$A \cap \overline{B} = \emptyset \wedge \overline{A} \cap B = \emptyset.$$

Definition 3.3.56 *A subset E of a metric space X is* ***connected*** *if and only if E is not the union of two nonempty separated sets.*

Example 3.3.57 *To justify that $A = \{x \in \mathbb{R} : 0 < x < 2 \vee 2 < x \leq 3\}$ is not connected, we just have to note that $B_1 = \{x \in \mathbb{R} : 0 < x < 2\}$ and $B_2 = \{x \in \mathbb{R} : 2 < x \leq 3\}$ are separated sets in $\mathbb{R}$ such that $A = B_1 \cup B_2$.*

Example 3.3.58 *In Euclidean 2-space, if $C = D_1 \cup D_2$ where*

$$D_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 : d\left((1,0), (x_1, x_2) \leq 1\right)\right\}$$

and

$$D_2 = \left\{(x_1, x_2) \in \mathbb{R}^2 : d\left((-1,0), (x_1, x_2) < 1\right)\right\},$$

then C is a connected subset of $\mathbb{R}^2$.

Remark 3.3.59 *The following is a symbolic description for a subset E of a metric space X to be connected:*

$$(\forall A)(\forall B)\left[(A \subset X \wedge B \subset X \wedge E = A \cup B) \Rightarrow \left(A \cap \overline{B} \neq \emptyset \vee \overline{A} \cap B \neq \emptyset \vee A = \emptyset \vee B = \emptyset\right)\right].$$

The statement is suggestive of the approach that is frequently taken when trying to prove sets having given properties are connected; namely, the direct approach would take an arbitrary set E and let $E = A \cup B$. This would be followed by using other information that is given to show that one of the sets must be empty.

The good news is that connected subsets of $\mathbb{R}^1$ can be characterized very nicely.

Theorem 3.3.60 *Let E be a subset of $\mathbb{R}^1$. Then E is connected in $\mathbb{R}^1$ if and only if*

$$(\forall x)\,(\forall y)\,(\forall z)\left[\left(x, y \in E \wedge z \in \mathbb{R}^1 \wedge x < z < y\right) \Rightarrow z \in E\right].$$

Excursion 3.3.61 *Fill in what is missing in order to complete the following proof of the Theorem.*

Proof. Suppose that E is a subset of $\mathbb{R}^1$ with the property that there exist real numbers x and y with $x < y$ such that $x, y \in E$ and, for some $z \in \mathbb{R}^1$,

$$z \in (x, y) \text{ and } z \notin E.$$

Let $A_z = E \cap (-\infty, z)$ and $B_z = E \cap (z, \infty)$. Since $z \notin E$, $E = A_z \cup B_z$. Because $x \in A_z$ and $y \in B_z$, both A_z and B_z are $\underset{(1)}{\underline{\hspace{3cm}}}$. Finally, $A_z \subset (-\infty, z)$ and $B_z \subset (z, \infty)$ yields that

$$\overline{A}_z \cap B_z = A_z \cap \overline{B}_z = \underset{(2)}{\underline{\hspace{1.5cm}}}.$$

Hence, E can be written as the union of two $\underset{(3)}{\underline{\hspace{3cm}}}$ sets; i.e., E is $\underset{(4)}{\underline{\hspace{6cm}}}$. Therefore, if E is connected, then $x, y \in E \wedge z \in \mathbb{R} \wedge x < z < y$ implies that $\underset{(5)}{\underline{\hspace{3cm}}}$.

To prove the converse, suppose that E is a subset of $\mathbb{R}^1$ that is not connected. Then there exist two nonempty separated subsets of $\mathbb{R}^1$, A and B, such that $E = A \cup B$. Choose $x \in A$ and $y \in B$ and assume that the set-up admits that $x < y$. Since $A \cap [x, y]$ is a nonempty subset of real numbers, by the least upper bound property, $z \underset{def}{=} \sup\left(A \cap [x, y]\right)$ exists and is real. From Theorem 3.3.26, $z \in \overline{A}$; then $\overline{A} \cap B = \emptyset$ yields that $z \notin B$. Now we have two possibilities to consider; $z \notin A$ and $z \in A$. If $z \notin A$, then $z \notin A \cup B = E$ and $x < z < y$. If $z \in A$, then $A \cap \overline{B} = \emptyset$ implies that $z \notin \overline{B}$ and we conclude that there exists w such that $z < w < y$ and $w \notin B$. From $z < w$, $w \notin A$. Hence, $w \notin A \cup B = E$ and $x < w < y$. In either case, we have that $\neg\,(\forall x)\,(\forall y)\,(\forall z)\left[\left(x, y \in E \wedge z \in \mathbb{R}^1 \wedge x < z < y\right) \Rightarrow z \in E\right]$. By the contrapositive $(\forall x)\,(\forall y)\,(\forall z)\left[\left(x, y \in E \wedge z \in \mathbb{R}^1 \wedge x < z < y\right) \Rightarrow z \in E\right]$ implies that E is connected. ■

Acceptable responses are: (1) nonempty, (2) $\emptyset$, (3) separated, (4) not connected, and (5) E is connected.

From the theorem, we know that, for a set of reals to be connected it must be either empty, all of $\mathbb{R}$, an interval, a segment, or a half open interval.

3.3.6 Perfect Sets

Definition 3.3.62 *A subset E of a metric space X is* ***perfect*** *if and only if E is closed and every point of E is a limit point of E.*

Alternatively, a subset E of a metric space X is perfect if and only if E is closed and contains no isolated points.

From Theorem 3.3.7, we know that any neighborhood of a limit point of a subset E of a metric space contains infinitely many points from E. Consequently, any nonempty perfect subset of a metric space is necessarily infinite; with the next theorem it is shown that, in Euclidean n-space, the nonempty perfect subsets are uncountably infinite.

Theorem 3.3.63 *If P is a nonempty perfect subset of Euclidean n-space, then P is uncountable.*

Proof. Let P be a nonempty perfect subset of $\mathbb{R}^n$. Then P contains at least one limit point and, by Theorem 3.3.6, P is infinite. Suppose that P is denumerable. It follows that P can be arranged as an infinite sequence; let

$$x_1, x_2, x_3, \ldots$$

represent the elements of P. First, we will justify the existence (or construction) of a sequence of neighborhoods $\left\{V_j\right\}_{j=1}^{\infty}$ that satisfies the following conditions:

(i) $(\forall j)\left(j \in \mathbb{J} \Rightarrow \overline{V}_{j+1} \subseteq V_j\right)$,

(ii) $(\forall j)\left(j \in \mathbb{J} \Rightarrow x_j \notin \overline{V}_{j+1}\right)$, and

(iii) $(\forall j)\left(j \in \mathbb{J} \Rightarrow V_j \cap P \neq \emptyset\right)$.

Start with an arbitrary neighborhood of x_1; i.e., let V_1 be any neighborhood of x_1. Suppose that $\left\{V_j\right\}_{j=1}^{n}$ has been constructed satisfying conditions (i)–(iii) for $1 \leq j \leq n$. Because P is perfect, every $x \in V_n \cap P$ is a limit point of P. Thus there are an infinite number of points of P that are in V_n and we may choose $y \in V_n \cap P$ such that $y \neq x_n$. Let V_{n+1} be a neighborhood of y such that $x_n \notin \overline{V}_{n+1}$ and $\overline{V}_{n+1} \subseteq V_n$. *Show that you can do this.*

Note that $V_{n+1} \cap P \neq \emptyset$ since $y \in V_{n+1} \cap P$. Thus we have a sequence $\{V_j\}_{j=1}^{n+1}$ satisfying (i)–(iii) for $1 \leq j \leq n+1$. By the Principle of Complete Induction we can construct the desired sequence.

Let $\{K_j\}_{j=1}^{\infty}$ be the sequence defined by $K_j = \overline{V_j} \cap P$ for each j. Since $\overline{V_j}$ and P are closed, K_j is closed. Since $\overline{V_j}$ is bounded, K_j is bounded. Thus K_j is closed and bounded and hence compact. Since $x_j \notin K_{j+1}$, no point of P lies in $\cap_{j=1}^{\infty} K_j$. Since $K_j \subseteq P$, this implies $\cap_{j=1}^{\infty} K_j = \emptyset$. But each K_j is nonempty by (iii) and $K_j \supseteq K_{j+1}$ by (i). This contradicts the Corollary 3.3.27. ■

Corollary 3.3.64 *For any two real numbers a and b such that $a < b$, the segment (a, b) is uncountable.*

The Cantor Set

The Cantor set is a fascinating example of a perfect subset of $\mathbb{R}^1$ that contains no segments. In Chapter 11 the idea of the measure of a set is studied; it generalizes the idea of length. If you take MAT127C, you will see the Cantor set offered as an example of a set that has measure zero even though it is uncountable.

The Cantor set is defined to be the intersection of a sequence of closed subsets of $[0, 1]$; the sequence of closed sets is defined recursively. Let $E_0 = [0, 1]$. For E_1 partition the interval E_0 into three subintervals of equal length and remove the middle segment (the interior of the middle section). Then

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

For E_2 partition each of the intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ into three subintervals of equal length and remove the middle segment from each of the partitioned intervals; then

$$\begin{aligned} E_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\ &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]. \end{aligned}$$

Continuing the process E_n will be the union of 2^n intervals. To obtain E_{n+1}, we partition each of the 2^n intervals into three subintervals of equal length and remove the middle segment, then E_{n+1} is the union of the 2^{n+1} intervals that remain.

Excursion 3.3.65 *In the space provided sketch pictures of E_0, E_1, E_2, and E_3 and find the sum of the lengths of the intervals that form each set.*

By construction $\{E_n\}_{n=1}^{\infty}$ is a nested sequence of compact subsets of $\mathbb{R}^1$.

Excursion 3.3.66 *Find a formula for the sum of the lengths of the intervals that* *form each set E_n.*

The Cantor set is defined to be $P = \bigcap_{n=1}^{\infty} E_n$.

Excursion 3.3.67 *Justify each of the following claims.*

(a) The Cantor set is compact.

(b) The $\{E_n\}_{n=1}^{\infty}$ satisfies the finite intersection property

Remark 3.3.68 *It follows from the second assertion that P is nonempty.*

Finally we want to justify the claims that were made about the Cantor set before we described its construction.

- *The Cantor set contains no segment from E_0.*

  To see this, we observe that each segment in the form of

  $$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \text{ for } k, m \in \mathbb{J}$$

  is disjoint from P. Given any segment (α, β) for $\alpha < \beta$, if $m \in \mathbb{J}$ is such that $3^{-m} < \dfrac{\beta - \alpha}{6}$, then (α, β) contains an interval of the form $\left(\dfrac{3k+1}{3^m}, \dfrac{3k+2}{3^m}\right)$ from which it follows that (α, β) is not contained in P.

- *The Cantor set is perfect.* For $x \in P$, let S be any segment that contains x. Since $x \in \bigcap_{n=1}^{\infty} E_n$, $x \in E_n$ for each $n \in \mathbb{J}$. Corresponding to each $n \in \mathbb{J}$, let I_n be the interval in E_n such that $x \in I_n$. Now, choose $m \in \mathbb{J}$ large enough to get $I_m \subset S$ and let x_m be an endpoint of I_m such that $x_m \neq x$. From the way that P was constructed, $x_m \in P$. Since S was arbitrary, we have shown that every segment containing x also contains at least one element from P. Hence, x is a limit point of P. That x was arbitrary yields that every element of P is a limit point of P.

3.4 Problem Set C

1. For $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $\mathbf{y} = (y_1, y_2, ..., y_N)$ in $\mathbb{R}^N$, let

   $$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{j=1}^{N} \left(x_j - y_j\right)^2}.$$

   Prove that $(\mathbb{R}^N, d)$ is a metric space.

2. For $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $\mathbf{y} = (y_1, y_2, ..., y_N)$ in $\mathbb{R}^N$, let
$$D(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{N} |x_j - y_j|.$$
Prove that $(\mathbb{R}^N, D)$ is a metric space.

3. For $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $\mathbf{y} = (y_1, y_2, ..., y_N)$ in $\mathbb{R}^N$, let
$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \le j \le N} \left| x_j - y_j \right|.$$
Prove that $(\mathbb{R}^N, d_\infty)$ is a metric space.

4. Show that the Euclidean metric d, given in problem #1, is equivalent to the metric d_∞, given in problem #3.

5. Suppose that (S, d) is a metric space. Prove that $\left(S, d'\right)$ is a metric space where
$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$
[Hint: You might find it helpful to make use of properties of $h(\xi) = \dfrac{\xi}{1 + \xi}$ for $\xi \geq 0$.]

6. If $a_1, a_2, ..., a_n$ are positive real numbers, is
$$d(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{n} a_k |x_k - y_k|$$
where $\mathbf{x} = (x_1, x_2, ..., x_n), \mathbf{y} = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, a metric on $\mathbb{R}^n$? Does your response change if the hypothesis is modified to require that $a_1, a_2, ..., a_n$ are nonnegative real numbers?

7. Is the metric D, given in problem #2, equivalent to the metric d_∞, given in problem #3? Carefully justify your position.

8. Are the metric spaces $(\mathbb{R}^N, d)$ and $\left(\mathbb{R}^N, d'\right)$ where the metrics d and d' are given in problems #1 and #5, respectively, equivalent? Carefully justify the position taken.

9. For (x_1, x_2) and $\left(x_1', x_2'\right)$ in $\mathbb{R}^2$,

$$d_3\left((x_1, x_2), \left(x_1', x_2'\right)\right) = \begin{cases} |x_2| + \left|x_2'\right| + \left|x_1 - x_1'\right| & \text{, if} \quad x_1 \neq x_1' \\ \left|x_2 - x_2'\right| & \text{, if} \quad x_1 = x_1' \end{cases}$$

   Show that $\left(\mathbb{R}^2, d_3\right)$ is a metric space.

10. For $x, y \in \mathbb{R}^1$, let $d(x, y) = |x - 3y|$. Is $(\mathbb{R}, d)$ a metric space? Briefly justify your position.

11. For $\mathbb{R}^1$ with $d(x, y) = |x - y|$, give an example of a set which is neither open nor closed.

12. Show that, in Euclidean $n - space$, a set that is open in $\mathbb{R}^n$ has no isolated points.

13. Show that every finite subset of $\mathbb{R}^N$ is closed.

14. For $\mathbb{R}^1$ with the Euclidean metric, let $A = \{x \in \mathbb{Q} : 0 \leq x \leq 1\}$. Describe $\overline{A}$.

15. Prove each of the following claims that are parts of Theorem 3.3.13. Let S be a metric space.

    (a) The union of any family $\mathcal{F}$ of open subsets of S is open.

    (b) The intersection of any family $\mathcal{F}$ of closed subsets of S is closed.

    (c) If $A_1, A_2, ..., A_m$ is a finite family of closed subsets of S, then the union $\bigcup_{j=1}^{m} A_j$ is closed.

    (d) The space S is both open and closed.

    (e) The null set is both open and closed.

16. For $X = [-8, -4) \cup \{-2, 0\} \cup \left(\mathbb{Q} \cap \left(1, 2\sqrt{2}\right]\right)$ as a subset of $\mathbb{R}^1$, identify (describe or show a picture of) each of the following.

    (a) The interior of X, $\operatorname{Int}(X)$

    (b) The exterior of X, $\operatorname{Ext}(X)$

    (c) The closure of X, $\overline{X}$

(d) The boundary of X, ∂X

(e) The set of isolated points of X

(f) The set of lower bounds for X and the least upper bound of X, $\sup(X)$

17. As subsets of Euclidean 2-space, let

$$A = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \max\{|x_1 + 1|, |x_2|\} \le \frac{1}{2} \right\},$$

$B = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \max\{|x_1 + 1|, |x_2|\} \le 1 \right\}$ and

$$Y = \left\{ (x_1, x_2) \in \mathbb{R}^2 : (x_1, x_2) \in B - A \vee \left((x_1 - 1)^2 + x_2^2 < 1 \right) \right\}.$$

(a) Give a nicely labelled sketch of Y on a representation for the Cartesian coordinate plane.

(b) Give a nicely labelled sketch of the exterior of Y, $\operatorname{Ext}(Y)$, on a representation for the Cartesian coordinate plane.

(c) Is Y open? Briefly justify your response.

(d) Is Y closed? Briefly justify your response.

(e) Is Y connected? Briefly justify your response.

18. Justify each of the following claims that were made in the Remark following Definition 3.3.15

(a) If A is a subset of a metric space (S, d), then $\operatorname{Ext}(A) = \operatorname{Int}(A^c)$.

(b) If A is a subset of a metric space (S, d), then

$$x \in \partial A \Leftrightarrow (\forall N_r(x)) \left(N_r(x) \cap A \ne \emptyset \wedge N_r(x) \cap A^c \ne \emptyset \right).$$

19. For $\mathbb{R}^2$ with the Euclidean metric, show that the set

$$S = \left\{ (x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1 \right\}$$

is open. Describe each of S^0, S', ∂S, $\overline{S}$, and S^c.

20. Prove that $\left\{ (x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 < 1 \wedge 0 \le x_2 \le 1 \right\}$ is not compact.

21. Prove that $\mathbb{Q}$, the set of rationals in $\mathbb{R}^1$, is not a connected subset of $\mathbb{R}^1$.

22. Let $\mathcal{F}$ be any family of connected subsets of a metric space X such that any two members of $\mathcal{F}$ have a common point. Prove that $\bigcup_{F\in\mathcal{F}} F$ is connected.

23. Prove that if S is a connected subset of a metric space, then $\overline{S}$ is connected.

24. Prove that any interval $I \subset \mathbb{R}^1$ is a connected subset of $\mathbb{R}^1$.

25. Prove that if A is a connected set in a metric space and $A \subset B \subset \overline{A}$, then B is connected.

26. Let $\{F_n\}_{n=1}^{\infty}$ be a nested sequence of compact sets, each of which is connected. Prove that $\bigcap_{n=1}^{\infty} F_n$ is connected.

27. Give an example to show that the compactness of the sets F_k given in problem #26 is necessary; i.e., show that a nested sequence of closed connected sets would not have been enough to ensure a connected intersection.

Chapter 4

Sequences and Series–First View

Recall that, for any set A, a sequence of elements of A is a function $f : \mathbb{J} \to A$. Rather than using the notation $f(n)$ for the elements that have been selected from A, since the domain is always the natural numbers, we use the notational convention $a_n = f(n)$ and denote sequences in any of the following forms:

$$\{a_n\}_{n=1}^{\infty}, \qquad \{a_n\}_{n\in\mathbb{J}}, \qquad or \qquad a_1, a_2, a_3, a_4, \ldots.$$

This is the only time that we use the set bracket notation $\{\quad\}$ in a different context. The distinction is made in the way that the indexing is communicated . For $a_n = \alpha$, the $\{a_n\}_{n=1}^{\infty}$ is the constant sequence that "lists the term α infinitely often," $\alpha, \alpha, \alpha, \alpha, \ldots$; while $\{a_n : n \in \mathbb{J}\}$ is the set consisting of one element α. (When you read the last sentence, you should have come up with some version of "For 'a sub n' equal to α, the sequence of 'a sub n' for n going from one to infinity is the constant sequence that "lists the term α infinitely often," $\alpha, \alpha, \alpha, \ldots$; while the set consisting of 'a sub n' for n in the set of positive integers is the set consisting of one element α"; i.e., the point is that you should not have skipped over the $\{a_n\}_{n=1}^{\infty}$ and $\{a_n : n \in \mathbb{J}\}$.) Most of your previous experience with sequences has been with sequences of real numbers, like

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots,\ldots \qquad \left\{\frac{3}{n+1}\right\}_{n=1}^{\infty},$$

$$\left\{\frac{n^2+3n-5}{n+47}\right\}_{n=1}^{\infty}, \ \left\{\frac{n^3-1}{n^3+1}+(-1)^n\right\}_{n=1}^{\infty}, \ and \ \left\{\frac{\log n}{n}+\sin\left(\frac{n\pi}{8}\right)\right\}_{n=1}^{\infty}.$$

In this chapter, most of our sequences will be of elements in Euclidean n-space. In MAT127B, our second view will focus on sequence of functions.

As children, our first exposure to sequences was made in an effort to teach us to look for patterns or to develop an appreciation for patterns that occur naturally.

Excursion 4.0.1 *For each of the following, find a description for the general term as a function of* $n \in \mathbb{J}$ *that fits the terms that are given.*

1. $\frac{2}{5}, \frac{4}{7}, \frac{8}{9}, \frac{16}{11}, \frac{32}{13}, \frac{64}{15}, \ldots$

2. $1, \frac{3}{5}, 9, \frac{7}{9}, 81, \frac{11}{13}, 729, \ldots$

An equation that works for (1) is $(2^n)(2n+3)^{-1}$ while (2) needs a different formula for the odd terms and the even terms; one pair that works is $(2n-1)(2n+1)^{-1}$ for n even and 3^{n-1} when n is odd.

As part of the bigger picture, pattern recognition is important in areas of mathematics or the mathematical sciences that generate and study models of various phenomena. Of course, the models are of value when they allow for analysis and/or making projections. In this chapter, we seek to build a deeper mathematical understanding of sequences and series; primary attention is on properties associated with convergence. After preliminary work with sequences in arbitrary metric spaces, we will restrict our attention to sequences of real and complex numbers.

4.1 Sequences and Subsequences in Metric Spaces

If you recall the definition of convergence from your frosh calculus course, you might notice that the definition of a limit of a sequence of points in a metric space merely replaces the role formerly played by absolute value with its generalization, distance.

Definition 4.1.1 *Let $\{p_n\}_{n=1}^{\infty}$ denote a sequence of elements of a metric space (S, d) and p_0 be an element of S. The* ***limit of*** *$\{p_n\}_{n=1}^{\infty}$ is p_0 as n tends to (goes to or approaches) infinity if and only if*

$$(\forall\varepsilon)\left[(\varepsilon \in \mathbb{R} \wedge \varepsilon > 0) \Rightarrow (\exists M = M(\varepsilon))\,(M \in \mathbb{J} \wedge (\forall n)\,(n > M \Rightarrow d(p_n, p_0) < \varepsilon))\right]$$

We write either $p_n \to p_0$ or $\lim\limits_{n\to\infty} p_n = p_0$.

Remark 4.1.2 *The description $M = M(\varepsilon)$ indicates that “limit of sequence proofs” require justification or specification of a means of prescribing how to find an M that “will work” corresponding to each $\varepsilon > 0$. A function that gives us a nice way to specify $M(\varepsilon)$’s is defined by*

$$\lceil x \rceil = \inf\{j \in \mathbb{Z} : x \leq j\}$$

and is sometimes referred to as the ***ceiling function****. Note, for example, that $\left\lceil \frac{1}{2} \right\rceil = 1$, $\lceil -2.2 \rceil = -2$, and $\lceil 5 \rceil = 5$. Compare this to the greatest integer function, which is sometimes referred to as the* ***floor function****.*

Example 4.1.3 *The sequence $\left\{\frac{2}{n}\right\}_{n=1}^{\infty}$ has the limit 0 in $\mathbb{R}$. We can take $M(1) = 2$, $M\left(\frac{1}{100}\right) = 200$, and $M\left(\frac{3}{350}\right) = \left\lceil \frac{700}{3} \right\rceil = 234$. Of course, three examples does not a proof make. In general, for $\varepsilon > 0$, let $M(\varepsilon) = \left\lceil \frac{2}{\varepsilon} \right\rceil$. Then $n > M(\varepsilon)$ implies that*

$$n > \left\lceil \frac{2}{\varepsilon} \right\rceil \geq \frac{2}{\varepsilon} > 0$$

which, by Proposition 1.2.9 (#7) and (#5), implies that $\frac{1}{n} < \frac{\varepsilon}{2}$ and $\frac{2}{n} = \left|\frac{2}{n}\right| < \varepsilon$.

Using the definition to prove that the limit of a sequence is some point in the metric space is an example of where our scratch work towards finding a proof might be quite different from the proof that is presented. This is because we are allowed to "work backwards" with our scratch work, but we can not present a proof that starts with the bound that we want to prove. We illustrate this with the following excursion.

Excursion 4.1.4 *After reading the presented scratch work, fill in what is missing to complete the proof of the claim that* $\left\{\frac{1+in}{n+1}\right\}_{n=1}^{\infty}$ *converges to* i *in* $\mathbb{C}$.

(a) ***Scratch work towards a proof****. Because* $i \in \mathbb{C}$, *it suffices to show that* $\lim_{n\to\infty} \frac{1+in}{n+1} = i$. *Suppose* $\varepsilon > 0$ *is given. Then*

$$\left|\frac{1+in}{n+1} - i\right| = \left|\frac{1+in-i\,(n+1)}{n+1}\right| = \left|\frac{1-i}{n+1}\right| = \frac{\sqrt{2}}{n+1} < \frac{\sqrt{2}}{n} < \varepsilon$$

whenever $\frac{\sqrt{2}}{\varepsilon} < n$. *So taking* $M(\varepsilon) = \left\lceil \frac{\sqrt{2}}{\varepsilon} \right\rceil$ *will work.*

(b) ***A proof.*** *For* $\varepsilon > 0$, *let* $M(\varepsilon) = \underset{(1)}{\underline{\qquad\qquad\qquad}}$. *Then* $n \in \mathbb{J}$ *and* $n > M(\varepsilon)$ *implies that* $n > \frac{\sqrt{2}}{\varepsilon}$ *which is equivalent to* $\underset{(2)}{\underline{\qquad\qquad\qquad}} < \varepsilon$. *Because* $n+1 > n$ *and* $\sqrt{2} > 0$, *we also know that* $\frac{\sqrt{2}}{\underset{(3)}{\underline{\qquad}}} < \frac{\overset{(4)}{\underline{\qquad}}}{n}$. *Consequently, if* $n > M(\varepsilon)$, *then*

$$\left|\frac{1+in}{n+1} - i\right| = \left|\frac{1+in-i\,(n+1)}{n+1}\right| = \left|\frac{1-i}{n+1}\right| = \frac{\sqrt{2}}{n+1} < \underset{(5)}{\underline{\qquad\qquad\qquad}} < \varepsilon$$

Since $\varepsilon > 0$ *was arbitrary, we conclude that*

$$(\forall\varepsilon)\left[(\varepsilon > 0) \Rightarrow (\exists M(\varepsilon))\left(M \in \mathbb{J} \wedge (\forall n)\left(n > M \Rightarrow \left|\frac{1+in}{n+1} - i\right| < \varepsilon\right)\right)\right];$$

i.e., $\underset{(6)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}$. *Finally,* $i = (0,1) \in \mathbb{C}$ *and* $\lim_{n\to\infty} \frac{1+in}{n+1} = i$ *yields that* $\left\{\frac{1+in}{n+1}\right\}_{n=1}^{\infty}$ *converges to* i *in* $\mathbb{C}$.

Acceptable responses are (1) $\left\lceil \frac{\sqrt{2}}{\varepsilon} \right\rceil$, (2) $\frac{\sqrt{2}}{n}$, (3) $n+1$, (4) $\sqrt{2}$, (5) $\frac{\sqrt{2}}{n}$, (6) $\lim_{n\to\infty} \frac{1+in}{n+1} = i$.

Definition 4.1.5 *The sequence $\{p_n\}_{n=1}^{\infty}$ of elements in a metric space S is said to* ***converge (****or* ***be convergent) in*** *S if there is a point $p_0 \in S$ such that $\lim_{n\to\infty} p_n = p_0$; it is said to* ***diverge in*** *S if it does not converge in S.*

Remark 4.1.6 *Notice that a sequence in a metric space S will be divergent in S if its limit is a point that is not in S. In our previous example, we proved that $\left\{\frac{2}{n}\right\}_{n=1}^{\infty}$ converges to 0 in $\mathbb{R}$; consequently, $\left\{\frac{2}{n}\right\}_{n=1}^{\infty}$ is convergent in Euclidean 1-space. On the other hand, $\left\{\frac{2}{n}\right\}_{n=1}^{\infty}$ is divergent in $\left(\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}, d\right)$ where d denotes the Euclidean metric on $\mathbb{R}$, $d(x,y) = |x-y|$.*

Our first result concerning convergent sequences is metric spaces assures us of the uniqueness of the limits when they exist.

Lemma 4.1.7 *Suppose $\{p_n\}_{n=1}^{\infty}$ is a sequence of elements in a metric space (S,d). Then*

$$(\forall p)(\forall q)\left(\left[p, q \in S \wedge \lim_{n\to\infty} p_n = p \wedge \lim_{n\to\infty} p_n = q\right] \Rightarrow q = p\right).$$

Space for scratch work.

Excursion 4.1.8 *Fill in what is missing in order to complete the proof of the lemma.*

Proof. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of elements in a metric space (S,d) for which there exists p and q in S such that $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} p_n = q$. Suppose the $p \neq q$. Then $d(p,q) > 0$ and we let $\varepsilon = \frac{1}{2} d(p,q)$. Because $\lim_{n\to\infty} p_n = p$ and $\varepsilon > 0$, there exists a positive integer M_1 such that

$$n > M_1 \Rightarrow d(p_n, p) < \varepsilon;$$

similarly, $\lim_{n\to\infty} p_n = q$ yields the existence of a positive integer M_2 such that

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{(1)}.$$

Now, let $M = \max\{M_1, M_2\}$. It follows from the symmetry property and the triangular inequality for metrics that $n > M$ implies that

$$d(p,q) \leq d(p,p_n) + \underbrace{\qquad\quad}_{(2)} < \varepsilon + \varepsilon = 2\left(\underbrace{\qquad\quad}_{(3)}\right) = d(p,q)$$

which contradicts the trichotomy law. Since we have reached a contradiction, we conclude that $\underbrace{\qquad\quad}_{(4)}$ as needed. Therefore, the limit of any convergent sequence in a metric space is unique. ■

Acceptable fill-ins are: (1) $n > M_2 \Rightarrow d(p_n, q) < \varepsilon$, (2) $d(p_n, q)$ (3) $\frac{1}{2} d(p,q)$, (4) $p = q$.

Definition 4.1.9 *The sequence $\{p_n\}_{n=1}^{\infty}$ of elements in a metric space (S,d) is* ***bounded*** *if and only if*

$$(\exists M)(\exists x)\left[M > 0 \wedge x \in S \wedge (\forall n)(n \in \mathbb{J} \Rightarrow d(x, p_n) < M)\right].$$

Note that if the sequence $\{p_n\}_{n=1}^{\infty}$ of elements in a metric space S is an not bounded, then the sequence is divergent in S. On the other hand, our next result shows that convergence yields boundedness.

Lemma 4.1.10 *If the sequence $\{p_n\}_{n=1}^{\infty}$ of elements in a metric space (S,d) is convergent in S, then it is bounded.*

Space for scratch work.

Proof. Suppose that $\{p_n\}_{n=1}^{\infty}$ is a sequence of elements in a metric space (S, d) that is convergent to $p_0 \in S$. Then, for $\varepsilon = 1$, there exists a positive integer $M = M(1)$ such that

$$n > M \Rightarrow d(p_n, p_0) < 1.$$

Because $\left\{d\left(p_j, p_0\right) : j \in \mathbb{J} \wedge 1 \leq j \leq M\right\}$ is a finite set of nonnegative real numbers, it has a largest element. Let

$$K = \max\left\{1, \max\left\{d\left(p_j, p_0\right) : j \in \mathbb{J} \wedge 1 \leq j \leq M\right\}\right\}.$$

Since $d(p_n, p_0) \leq K$, for each $n \in \mathbb{J}$, we conclude that $\{p_n\}_{n=1}^{\infty}$ is bounded. ■

Remark 4.1.11 *To see that the converse of Lemma 4.1.10 is false, for $n \in \mathbb{J}$, let*

$$p_n = \begin{cases} \dfrac{1}{n^2} & , \ \text{if } 2 \mid n \\[2ex] 1 - \dfrac{1}{n+3} & , \ \text{if } 2 \nmid n \end{cases}.$$

Then, for d the Euclidean metric on $\mathbb{R}^1$, $d(0, p_n) = |0 - p_n| < 1$ for all $n \in \mathbb{J}$, but $\{p_n\}_{n=1}^{\infty}$ is not convergent in $\mathbb{R}$.

Excursion 4.1.12 *For each $n \in \mathbb{J}$, let $a_n = p_{2n}$ and $b_n = p_{2n-1}$ where p_n is defined in Remark 4.1.11.*

(a) Use the definition to prove that $\lim_{n\to\infty} a_n = 0$.

(b) Use the definition to prove that $\lim_{n\to\infty} b_n = 1$.

***Note that $a_n = \dfrac{1}{(2n)^2}$ and $b_n = 1 - \dfrac{1}{(2n-1)+3} = 1 - \dfrac{1}{2(n+1)}$; if you used $\dfrac{1}{n^2}$ and $1 - \dfrac{1}{n+3}$, respectively, your choices for corresponding $M(\varepsilon)$ will be slightly off. The following are acceptable solutions, which of course are not unique; compare what you did for general sense and content. Make especially certain that you did not offer a proof that is "working backwards" from what you wanted to show. (a) For $\varepsilon > 0$, let $M = M(\varepsilon) = \left\lceil \dfrac{1}{2\sqrt{\varepsilon}} \right\rceil$. Then $n > M$ implies that $n > \left(2\sqrt{\varepsilon}\right)^{-1}$ or $\dfrac{1}{2n} < \sqrt{\varepsilon}$. If follows that $\left| \dfrac{1}{(2n)^2} - 0 \right| = \dfrac{1}{(2n)^2} = \dfrac{1}{2n} \cdot \dfrac{1}{2n} < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon$ whenever $n > M$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{1}{(2n)^2} = 0$. (b) For $\varepsilon > 0$, let $M = M(\varepsilon) = \left\lceil \dfrac{1}{\varepsilon} \right\rceil$. Then $n > M$ implies that $n > (\varepsilon)^{-1}$ or $\dfrac{1}{n} < \varepsilon$. Note that, for $n \in \mathbb{J}$, $n \geq 1 > 0$ implies that $n + 2 > 0 + 2 = 2 > 0$ and $2n + 2 = n + (n+2) > 0 + n = n$. Thus, for $n \in \mathbb{J}$ and $n > M$, we have that

$$\left| \left(1 - \frac{1}{2n+2}\right) - 1 \right| = \left| \frac{1}{2n+2} \right| = \frac{1}{2n+2} < \frac{1}{n} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(1 - \dfrac{1}{2n+2}\right) = 1$.***

Remark 4.1.13 *Hopefully, you spotted that there were some extra steps exhibited in our solutions to Excursion 4.1.12. I chose to show some of the extra steps that* *illustrated where we make explicit use of the ordered field properties that were discussed in Chapter 1. In particular, it is unnecessary for you to have explicitly demonstrated that* $2n + 2 > n$ *from the inequalities that were given in Proposition 1.2.9 or the step* $\frac{1}{2n} \cdot \frac{1}{2n} < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon}$ *that was shown in part (a). For the former you can just write things like* $2n + 2 > n$*; for the latter, you could just have written* $\frac{1}{(2n)^2} < \left(\sqrt{\varepsilon}\right)^2 = \varepsilon$.

What we just proved about the sequence given in Remark 4.1.11 can be translated to a statement involving subsequences.

Definition 4.1.14 *Given a sequence* $\{p_n\}_{n=1}^{\infty}$ *of elements in a metric space* X *and a sequence* $\{n_k\}_{k=1}^{\infty}$ *of positive integers such that of* $n_k < n_{k+1}$ *for each* $k \in \mathbb{J}$*, the sequence* $\left\{p_{n_k}\right\}_{k=1}^{\infty}$ *is called a* ***subsequence*** *of* $\{p_n\}_{n=1}^{\infty}$*. If* $\left\{p_{n_k}\right\}_{k=1}^{\infty}$ *converges in* X *then its limit is called a* ***subsequential limit*** *of* $\{p_n\}_{n=1}^{\infty}$.

Remark 4.1.15 *In our function terminology, a subsequence of* $f : \mathbb{J} \to X$ *is the restriction of* f *to any infinite subset of* $\mathbb{J}$ *with the understanding that ordering is conveyed by the subscripts; i.e.,* $n_j < n_{j+1}$ *for each* $j \in \mathbb{J}$.

From Excursion 4.1.12, we know that the sequence $\{p_n\}_{n=1}^{\infty}$ given in Remark 4.1.11 has two subsequential limits; namely, 0 and 1. The uniqueness of the limit of a convergent sequence leads us to observe that every subsequence of a convergent sequence must also be convergent to the same limit as the original sequence. Consequently, the existence of two distinct subsequential limits for $\{p_n\}_{n=1}^{\infty}$ is an alternative means of claiming that $\{p_n\}_{n=1}^{\infty}$ is divergent. In fact, it follows from the definition of the limit of a sequence that infinitely many terms outside of any neighborhood of a point in the metric space from which the sequence is chosen will eliminate that point as a possible limit. A slight variation of this observation is offered in the following characterization of convergence of a sequence in metric space.

Lemma 4.1.16 *Let* $\{p_n\}_{n=1}^{\infty}$ *be a sequence of elements from a metric space* (X, d)*. Then* $\{p_n\}_{n=1}^{\infty}$ *converges to* $p \in X$ *if and only if every neighborhood of* p *contains all but finitely many of the terms of* $\{p_n\}_{n=1}^{\infty}$.

Space for scratch work.

Proof. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of elements from a metric space (X, d).

Suppose that $\{p_n\}_{n=1}^{\infty}$ converges to $p \in X$ and V is a neighborhood of p. Then there exists a positive real number r such that $V = N_r(p)$. From the definition of a limit, there exists a positive integer $M = M(r)$ such that $n > M$ implies that $d(p, p_n) < r$; i.e., for all $n > M$, $p_n \in V$. Consequently, at most the $\{p_k : k \in \mathbb{J} \wedge 1 \leq k \leq M\}$ is excluded from V. Since V was arbitrary, we conclude that every neighborhood of p contains all but finitely many of the terms of $\{p_n\}_{n=1}^{\infty}$.

Suppose that every neighborhood of p contains all but finitely many of the terms of $\{p_n\}_{n=1}^{\infty}$. For any $\varepsilon > 0$, $N_\varepsilon(p)$ contains all but finitely many of the terms of $\{p_n\}_{n=1}^{\infty}$. Let $M = \max\{k \in \mathbb{J} : p_k \notin N_\varepsilon(p)\}$. Then $n > M$ implies that $p_n \in N_\varepsilon(p)$ from which it follows that $d(p_n, p) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that, for every $\varepsilon > 0$ there exists a positive integer $M = M(\varepsilon)$ such that $n > M$ implies that $d(p_n, p) < \varepsilon$; that is, $\{p_n\}_{n=1}^{\infty}$ converges to $p \in X$. ■

It will come as no surprise that limit point of subsets of metric spaces can be related to the concept of a limit of a sequence. The approach used in the proof of the next theorem should look familiar.

Theorem 4.1.17 *A point p_0 is a limit point of a subset A of a metric space (X, d) if and only if there is a sequence $\{p_n\}_{n=1}^{\infty}$ with $p_n \in A$ and $p_n \neq p_0$ for every n such that $p_n \to p_0$ as $n \to \infty$.*

Proof. $(\Leftarrow)$ Suppose that there is a sequence $\{p_n\}_{n=1}^{\infty}$ such that $p_n \in A$, $p_n \neq p_0$ for every n, and $p_n \to p_0$. For $r > 0$, consider the neighborhood $N_r(p_0)$. Since $p_n \to p_0$, there exists a positive integer M such that $d(p_n, p_0) < r$ for all $n > M$. In particular, $p_{M+1} \in A \cap N_r(p_0)$ and $p_{M+1} \neq p_0$. Since $r > 0$ was arbitrary, we conclude that p_0 is a limit point of the set A.

$(\Rightarrow)$ Suppose that $p_0 \in X$ is a limit point of A. (Finish this part by first making judicious use of the real sequence $\left\{\dfrac{1}{j}\right\}_{j=1}^{\infty}$ to generate a useful sequence $\{p_n\}_{n=1}^{\infty}$

followed by using the fact that $\frac{1}{j} \to 0$ as $j \to \infty$ to show that $\{p_n\}_{n=1}^{\infty}$ converges to p_0.)

■

Remark 4.1.18 *Since Theorem 4.1.17 is a characterization for limit points, it gives us an alternative definition for such. When called upon to prove things related to limit points, it can be advantageous to think about which description of limit points would be most fruitful; i.e., you can use the definition or the characterization interchangeably.*

We close this section with two results that relate sequences with the metric space properties of being closed or being compact.

Theorem 4.1.19 *If $\{p_n\}_{n=1}^{\infty}$ is a sequence in X and X is a compact subset of a metric space (S, d), then there exists a subsequence of $\{p_n\}_{n=1}^{\infty}$ that is convergent in X.*

Space for scratch work.

Proof. Suppose that $\{p_n\}_{n=1}^{\infty}$ is a sequence in X and X is a compact subset of a metric space (S, d). Let $P = \{p_n : n \in \mathbb{J}\}$. If P is finite, then there is at least one k such that $p_k \in P$ and, for infinitely many $j \in \mathbb{J}$, we have that $p_j = p_k$.

Consequently, we can choose a sequence $\left\{n_j\right\}_{j=1}^{\infty}$ such that $n_j < n_{j+1}$ and $p_{n_j} \equiv p_k$ for each $j \in \mathbb{J}$. It follows that $\left\{p_{n_j}\right\}_{j=1}^{\infty}$ is a (constant) subsequence of $\{p_n\}_{n=1}^{\infty}$ that is convergent to $p_k \in X$. If P is infinite, then P is an infinite subset of a compact set. By Theorem 3.3.46, it follows that P has a limit point p_0 in X. From Theorem 4.1.17, we conclude that there is a sequence $\{q_k\}_{k=1}^{\infty}$ with $q_k \in P$ and $q_k \neq p_0$ for every k such that $q_k \to p_0$ as $k \to \infty$; that is, $\{q_k\}_{k=1}^{\infty}$ is a subsequence of $\{p_n\}_{n=1}^{\infty}$ that is convergent to $p_0 \in X$. ■

Theorem 4.1.20 *If $\{p_n\}_{n=1}^{\infty}$ is a sequence in a metric space (S, d), then the set of all subsequential limits of $\{p_n\}_{n=1}^{\infty}$ is a closed subset of S.*

Space for scratch work.

Proof. Let E^* denote the set of all subsequential limits of the sequence $\{p_n\}_{n=1}^{\infty}$ of elements in the metric space (S, d). If E^* is finite, then it is closed. Thus, we can assume that E^* is infinite. Suppose that w is a limit point of E^*. Then, corresponding to $r = 1$, there exists $x \neq w$ such that $x \in N_1(w) \cap E^*$. Since $x \in E^*$, we can find a subsequence of $\{p_n\}_{n=1}^{\infty}$ that converges to x. Hence, we can choose $n_1 \in \mathbb{J}$ such that $p_{n_1} \neq w$ and $d\left(p_{n_1}, w\right) < 1$. Let $\delta = d\left(p_{n_1}, w\right)$. Because $\delta > 0$, w is a limit point of E^*, and E^* is infinite, there exists $y \neq w$ that is in $N_{\delta/4}(w) \cap E^*$. Again, $y \in E^*$ leads to the existence of a subsequence of $\{p_n\}_{n=1}^{\infty}$ that converges to y. This allows us to choose $n_2 \in \mathbb{J}$ such that $n_2 > n_1$ and $d\left(p_{n_2}, y\right) < \dfrac{\delta}{4}$. From the triangular inequality, $d\left(w, p_{n_2}\right) \leq d(w, y) + d\left(y, p_{n_2}\right) < \dfrac{\delta}{2}$. We can repeat this process. In general, if we have chosen the increasing finite sequence $n_1, n_2, ..., n_j$, then there exists a u such that $u \neq w$ and $u \in N_{r_j}(w) \cap E^*$ where $r_j = \dfrac{\delta}{2^{j+1}}$. Since $u \in E^*$, u is the limit of a subsequence of $\{p_n\}_{n=1}^{\infty}$. Thus, we can find n_{j+1} such that $d\left(p_{n_{j+1}}, u\right) < r_j$ from which it follows that

$$d\left(w, p_{n_{j+1}}\right) \leq d(w, u) + d\left(u, p_{n_{j+1}}\right) < 2r_j = \frac{\delta}{2^j}.$$

The method of selection of the subsequence $\left\{p_{n_j}\right\}_{j=1}^{\infty}$ ensures that it converges to w. Therefore, $w \in E^*$. Because w was arbitrary, we conclude that E^* contains all of its limit points; i.e., E^* is closed. ■

4.2 Cauchy Sequences in Metric Spaces

The following view of "proximity" of terms in a sequence doesn't isolate a point to serve as a limit.

Definition 4.2.1 *Let* $\{p_n\}_{n=1}^{\infty}$ *be an infinite sequence in a metric space* (S, d)*. Then* $\{p_n\}_{n=1}^{\infty}$ *is said to be a* ***Cauchy sequence*** *if and only if*

$$(\forall \varepsilon)\left[\varepsilon > 0 \Rightarrow (\exists M = M(\varepsilon))\,(M \in \mathbb{J} \wedge (\forall m)\,(\forall n)\,(n, m > M \Rightarrow d(p_n, p_m) < \varepsilon))\right].$$

Another useful property of subsets of a metric space is the diameter. In this section, the term leads to a characterization of Cauchy sequences as well as a sufficient condition to ensure that the intersection of a sequence of nested compact sets will consist of exactly one element.

Definition 4.2.2 *Let* E *be a subset of a metric space* (X, d)*. Then the* ***diameter of*** E*, denoted by* $\operatorname{diam}(E)$ *is*

$$\sup\{d(p, q) : p \in E \wedge q \in E\}.$$

Example 4.2.3 *Let* $A = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\right\}$ *and*

$$B = \left\{(x_1, x_2) \in \mathbb{R}^2 : \max\{|x_1|, |x_2|\} \leq 1\right\}.$$

Then, in Euclidean 2*-space,* $\operatorname{diam}(A) = 2$ *and* $\operatorname{diam}(B) = 2\sqrt{2}$.

Note that, for the sets A and B given in Example 4.2.3, $\operatorname{diam}\left(\overline{A}\right) = 2 = \operatorname{diam}(A)$ and $\operatorname{diam}\left(\overline{B}\right) = 2\sqrt{2} = \operatorname{diam}(B)$. These illustrate the observation that is made with the next result.

Lemma 4.2.4 *If E is any subset of a metric space X, then* $\operatorname{diam}(E) = \operatorname{diam}\left(\overline{E}\right)$.

Excursion 4.2.5 *Use the space provided to fill in a proof of the lemma. (If you get stuck, a proof can be found on page 53 of our text.*

The property of being a Cauchy sequence can be characterized nicely in terms of the diameter of particular subsequence.

Lemma 4.2.6 *If $\{p_n\}_{n=1}^{\infty}$ is an infinite sequence in a metric space X and E_M is the subsequence $p_M, p_{M+1}, p_{M+2}, \ldots$, then $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence if and only if* $\lim_{M\to\infty} \operatorname{diam}(E_M) = 0$.

Proof. Corresponding to the infinite sequence $\{p_n\}_{n=1}^{\infty}$ in a metric space (X, d) let E_M denote the subsequence $p_M, p_{M+1}, p_{M+2}, \ldots$.

Suppose that $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence. For $j \in \mathbb{J}$, there exists a positive integer $M_j^* = M_j^*(\varepsilon)$ such that $n, m > M_j^*$ implies that $d(p_n, p_m) < \dfrac{1}{j}$. Let $M_j = M_j^* + 1$. Then, for any $u, v \in E_{M_j}$, it follows that $d(u, v) < \dfrac{1}{j}$. Hence, $\sup\left\{d(u, v) : u \in E_{M_j} \wedge v \in E_{M_j}\right\} \leq \dfrac{1}{j}$; i.e., $\operatorname{diam}\left(E_{M_j}\right) \leq \dfrac{1}{j}$. Now given any $\varepsilon > 0$, there exists M' such that $j > M'$ implies that $\frac{1}{j} < \varepsilon$. For $M = \max\left\{M_j, M'\right\}$ and $j > M$, $\operatorname{diam}\left(E_{M_j}\right) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{M\to\infty} \operatorname{diam}(E_M) = 0$.

Suppose that $\lim_{M\to\infty} \operatorname{diam}(E_M) = 0$ and let $\varepsilon > 0$. Then there exists a positive integer K such that $m > K$ implies that $\operatorname{diam}(E_m) < \varepsilon$; i.e.,

$$\sup\{d(u,v) : u \in E_m \wedge v \in E_m\} < \varepsilon.$$

In particular, for $n, j > m$ we can write $n = m + x$ and $j = m + y$ for some positive integers x and y and it follows that

$$d\left(p_n, p_j\right) \leq \sup\{d(u,v) : u \in E_m \wedge v \in E_m\} < \varepsilon.$$

Thus, we have shown that, for any $\varepsilon > 0$, there exists a positive integer m such that $n, j > m$ implies that $d\left(p_n, p_j\right) < \varepsilon$. Therefore, $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence. ■

With Corollary 3.3.44, we saw that any nested sequence of nonempty compact sets has nonempty intersection. The following slight modification results from adding the hypothesis that the diameters of the sets shrink to 0.

Theorem 4.2.7 *If $\{K_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty compact subsets of a metric space X such that*

$$\lim_{n\to\infty} \operatorname{diam}(K_n) = 0,$$

then $\bigcap_{n\in\mathbb{J}} K_n$ consists of exactly one point.

Space for scratch work.

Proof. Suppose that $\{K_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty compact subsets of a metric space (X,d) such that $\lim_{n\to\infty} \operatorname{diam}(K_n) = 0$. From Corollary 3.3.44, $\{K_n\}_{n=1}^{\infty}$ being a nested sequence of nonempty compact subsets implies that $\bigcap_{n\in\mathbb{J}} K_n \neq \emptyset$.

If $\bigcap_{n\in\mathbb{J}} K_n$ consists of more that one point, then there exists points x and y in X such that $x \in \bigcap_{n\in\mathbb{J}} K_n$, $y \in \bigcap_{n\in\mathbb{J}} K_n$ and $x \neq y$. But this yields that

$$0 < d\,(x, y) \leq \sup\{d\,(p, q) : p \in K_n \wedge q \in K_n\}$$

for all $n \in \mathbb{J}$; i.e., $\operatorname{diam}(E_M) \nless d\,(x, y)$ for any $M \in \mathbb{J}$. Hence, $\lim_{n\to\infty} \operatorname{diam}(E_M) \neq 0$. Because $\lim_{n\to\infty} \operatorname{diam}(K_n) = 0$, it follows immediately that $\bigcap_{n\in\mathbb{J}} K_n$ consists of exactly one point. ■

Remark 4.2.8 *To see that a Cauchy sequence in an arbitrary metric space need not converge to a point that is in the space, consider the metric space* (S, d) *where* S *is the set of rational numbers and* $d(a, b) = |a - b|$.

On the other hand, a sequence that is convergent in a metric space is Cauchy there.

Theorem 4.2.9 *Let* $\{p_n\}_{n=1}^{\infty}$ *be an infinite sequence in a metric space* (S, d). *If* $\{p_n\}_{n=1}^{\infty}$ *converges in* S, *then* $\{p_n\}_{n=1}^{\infty}$ *is Cauchy.*

Proof. Let $\{p_n\}_{n=1}^{\infty}$ be an infinite sequence in a metric space (S, d) that converges in S to p_0. Suppose $\varepsilon > 0$ is given. Then, there exists an $M \in \mathbb{J}$ such that $n > M \Rightarrow d\,(p_n, p_0) < \frac{\varepsilon}{2}$. From the triangular inequality, if $n > M$ and $m > M$, then

$$d\,(p_n, p_m) \leq d\,(p_n, p_0) + d\,(p_0, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\{p_n\}_{n=1}^{\infty}$ is Cauchy. ■

As noted by Remark 4.2.8, the converse of Theorem 4.2.9 is not true. However, if we restrict ourselves to sequences of elements from compact subsets of a metric space, we obtain the following partial converse. Before showing this, we will make some us

Theorem 4.2.10 *Let* A *be a compact subset of a metric space* (S, d) *and* $\{p_n\}_{n=1}^{\infty}$ *be a sequence in* A. *If* $\{p_n\}_{n=1}^{\infty}$ *is Cauchy, then there exists a point* $p_0 \in A$ *such that* $p_n \to p_0$ *as* $n \to \infty$.

Proof. Let A be a compact subset of a metric space (S, d) and suppose that $\{p_n\}_{n=1}^{\infty}$ of elements in A is Cauchy. Let E_M be the subsequence $\left\{p_{M+j}\right\}_{j=0}^{\infty}$. Then $\left\{\overline{E}_M\right\}_{m=1}^{\infty}$ is a nested sequence of closed subsets of A and $\left\{\overline{E}_M \cap A\right\}_{m=1}^{\infty}$ is a nested sequence of compact subsets of S for which $\lim_{M\to\infty} \operatorname{diam}\left(\overline{E}_M \cap A\right) = 0$. By Theorem 4.2.7, there exists a unique p such that $p \in \overline{E}_M \cap A$ for all M.

Now justify that $p_n \to p$ as $n \to \infty$.

■

4.3 Sequences in Euclidean k-space

When we restrict ourselves to Euclidean space we get several additional results including the equivalence of sequence convergence with being a Cauchy sequence. The first result is the general version of the one for Euclidean n-space that we discussed in class.

Lemma 4.3.1 *On* $(\mathbb{R}^k, d)$*, where d denotes the Euclidean metric, let*

$$p_n = (x_{1n}, x_{2n}, x_{3n}, \ldots, x_{kn}).$$

Then the sequence $\{p_n\}_{n=1}^{\infty}$ *converges to* $P = (p_1, p_2, p_3, \ldots, p_k)$ *if and only if* $x_{jn} \to p_j$ *for each* j*,* $1 \le j \le k$ *as sequences in* $\mathbb{R}^1$.

Proof. The result follows from the fact that, for each m, $1 \le m \le k$,

$$|x_{mn} - p_m| = \sqrt{(x_{mn} - p_m)^2} \le \sqrt{\sum_{j=1}^{k} \left(x_{jn} - p_j\right)^2} \le \sum_{j=1}^{k} \left|x_{jn} - p_j\right|.$$

Suppose that $\varepsilon > 0$ is given. If $\{p_n\}_{n=1}^{\infty}$ converges to $P = (p_1, p_2, p_3, \ldots, p_k)$, then there exists a positive real number $M = M(\varepsilon)$ such than $n > M$ implies that

$$d(p_n, P) = \sqrt{\sum_{j=1}^{k} \left(x_{jn} - p_j\right)^2} < \varepsilon.$$

Hence, for each m, $1 \leq m \leq k$, and for all $n > M$,

$$|x_{mn} - p_m| \leq d(p_n, P) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{n\to\infty} x_{mn} = p_m$. Conversely, suppose that $x_{jn} \to p_j$ for each j, $1 \leq j \leq k$ as sequences in $\mathbb{R}^1$. Then, for each j, $1 \leq j \leq k$, there exists a positive integer $M_j = M_j(\varepsilon)$ such that $n > M_j$ implies that $|x_{mn} - p_m| < \dfrac{\varepsilon}{k}$. Let $M = \max_{1\leq j\leq k} M_j$. It follows that, for $n > M$,

$$d(p_n, P) \leq \sum_{j=1}^{k} \left|x_{jn} - p_j\right| < k\left(\frac{\varepsilon}{k}\right) = \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, we have that $\lim_{n\to\infty} p_n = P$. ■

Once we are restricted to the real field we can relate sequence behavior with algebraic operations involving terms of given sequences. The following result is one of the ones that allows us to find limits of given sequences from limits of sequences that we know or have already proved elsewhere.

Theorem 4.3.2 *Suppose that* $\{z_n\}_{n=1}^{\infty}$ *and* $\{\zeta_n\}_{n=1}^{\infty}$ *are sequences of complex numbers such that* $\lim_{n\to\infty} z_n = S$ *and* $\lim_{n\to\infty} \zeta_n = T$. *Then*

(a) $\lim_{n\to\infty} (z_n + \zeta_n) = S + T$;

(b) $\lim_{n\to\infty} (cz_n) = cS$, *for any constant* c;

(c) $\lim_{n\to\infty} (z_n\zeta_n) = ST$;

(d) $\lim_{n\to\infty} \left(\dfrac{z_n}{\zeta_n}\right) = \dfrac{S}{T}$, *provided that* $(\forall n)\left[n \in \mathbb{J} \Rightarrow \zeta_n \neq 0\right] \wedge T \neq 0$.

Excursion 4.3.3 *For each of the following, fill in either the proof in the box on the left of scratch work (notes) that support the proof that is given. If you get stuck, proofs can be found on pp 49-50 of our text.*

Proof. Suppose that $\{z_n\}_{n=1}^{\infty}$ and $\{\zeta_n\}_{n=1}^{\infty}$ are sequences of complex numbers such that $\lim_{n\to\infty} z_n = S$ and $\lim_{n\to\infty} \zeta_n = T$.

(a)		*Space for scratch work.* Need look at $\vert(z_n+\zeta_n)-(S+T)\vert$ –Know we can make $\vert z_m - S\vert < \frac{\varepsilon}{2}$ for $m > M_1$ & $\vert\zeta_n - T\vert < \frac{\varepsilon}{2}$ for $n > M_2$ –Go for $M = \max\{M_1, M_2\}$ and use Triangular Ineq.

(b)		*Space for scratch work.* Need look at $\vert(cz_n) - cS\vert = \vert c\vert\,\vert z_n - S\vert$ –Know we can make $\vert z_m - S\vert < \frac{\varepsilon}{\vert c\vert}$ for $m > M_1$ –for $c \neq 0$, mention $c = 0$ ——as separate case.

(c)	Since $z_n \to S$, there exists $M_1 \in \mathbb{J}$ such that $n > M_1$ implies that $\vert z_n - S\vert < 1$. Hence, $\vert z_n\vert - \vert S\vert < 1$ or $\vert z_n\vert < 1 + \vert S\vert$ for all $n > M_1$. Suppose that $\varepsilon > 0$ is given. If $T = 0$, then $\zeta_n \to 0$ as $n \to \infty$ implies that there exists $M^* \in \mathbb{J}$ such that $\vert\zeta_n\vert < \frac{\varepsilon}{1+\vert S\vert}$ whenever $n > M^*$. For $n > \max\{M_1, M^*\}$, it follows that $\vert(z_n\zeta_n) - ST\vert = \vert z_n\zeta_n\vert < (1+\vert S\vert)\left(\frac{\varepsilon}{1+\vert S\vert}\right) = \varepsilon.$ Thus, $\lim_{n\to\infty} z_n\zeta_n = 0$. If $T \neq 0$, then $\zeta_n \to T$ as $n \to \infty$ yields that there exists $M_2 \in \mathbb{J}$ such that $\vert\zeta_n\vert < \frac{\varepsilon}{2(1+\vert S\vert)}$ whenever $n > M_2$. From $z_n \to S$, there exists $M_3 \in \mathbb{J}$ such that $n > M_3 \Rightarrow$ $\vert z_n - T\vert < \frac{\varepsilon}{2\vert T\vert}$. Finally, for any $n > \max\{M_1, M_2, M_3\}$, $\vert(z_n\zeta_n) - ST\vert$ $= \vert(z_n\zeta_n) - z_nT + z_nT - ST\vert \leq \vert z_n\vert\,\vert\zeta_n - T\vert + \vert\zeta_n\vert\,\vert z_n - S\vert$ $< (1+\vert S\vert)\frac{\varepsilon}{2(1+\vert S\vert)} + \vert T\vert\frac{\varepsilon}{2\vert T\vert} = \varepsilon.$	*Space for* scratch work.

(d)

	Space for scratch work. $\left(\frac{z_n}{\zeta_n}\right) = z_n \left(\frac{1}{\zeta_n}\right)$ —we can just apply the result from (c).

■

The following result is a useful tool for proving the limits of given sequences in $\mathbb{R}^1$.

Lemma 4.3.4 (The Squeeze Principle) *Suppose that* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are sequences of real numbers such that* $\lim_{n\to\infty} x_n = S$ *and* $\lim_{n\to\infty} y_n = S$. *If* $\{u_n\}_{n=1}^{\infty}$ *is a sequence of real numbers such that, for some positive integer K*

$$x_n \le u_n \le y_n, \text{ for all } n > K,$$

then $\lim_{n\to\infty} u_n = S$.

Excursion 4.3.5 *Fill in a proof for The Squeeze Principle.*

Theorem 4.3.6 (Bolzano-Weierstrass Theorem) *In* $\mathbb{R}^k$, *every bounded sequence contains a convergent subsequence.*

Proof. Suppose that $\{p_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}^k$. Then $P \underset{def}{=} \{p_n : n \in \mathbb{J}\}$ is bounded. Since $\overline{P}$ is a closed and bounded subset of $\mathbb{R}^k$, by the Heine-Borel Theorem, $\overline{P}$ is compact. Because $\{p_n\}_{n=1}^{\infty}$ is a sequence in $\overline{P}$ a compact subset of a metric space, by Theorem 4.1.19, there exists a subsequence of $\{p_n\}_{n=1}^{\infty}$ that is convergent in $\overline{P}$. ■

Theorem 4.3.7 ($\mathbb{R}^k$ Completeness Theorem) *In $\mathbb{R}^k$, a sequence is convergent if and only if it is a Cauchy sequence.*

Excursion 4.3.8 *Fill in what is missing in order to complete the following proof of the $\mathbb{R}^k$ Completeness Theorem.*

Proof. Since we are in Euclidean k-space, by Theorem $\underset{(1)}{\underline{\qquad\qquad\qquad}}$, we know that any sequence that is convergent in $\mathbb{R}^k$ is a Cauchy sequence. Consequently, we only need to prove the converse.

Let $\{p_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^k$. Then corresponding to $\varepsilon = 1$, there exists $M = M(1) \in \mathbb{J}$ such that $m, n > M$ implies that $\underset{(2)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}$ where d denotes the Euclidean metric. In particular, $d(p_n, p_{M+1}) < 1$ for all $n > M$. Let

$$B = \max\left\{1, \max_{1 \le j \le M} d\left(p_j, d_{M+1}\right)\right\}.$$

Then, for each $j \in \mathbb{J}$, $d\left(p_j, d_{M+1}\right) \le B$ and we conclude that $\{p_n\}_{n=1}^{\infty}$ is a $\underset{(3)}{\underline{\qquad\qquad\qquad}}$ sequence in $\mathbb{R}^k$. From the $\underset{(4)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}$ Theorem, $\overline{\{p_n : n \in \mathbb{J}\}}$ is a compact subset of $\mathbb{R}^k$. Because $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a compact metric space, by Theorem 4.2.10, there exists a $p_0 \in \overline{\{p_n : n \in \mathbb{J}\}}$ such that $p_n \to p_0$ as $n \to \infty$. Since $\{p_n\}_{n=1}^{\infty}$ was arbitrary, we concluded that $\underset{(5)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}$. ■

Acceptable responses are: (1) 4.2.9, (2) $d(p_n, p_m) < 1$, (3) bounded, (4) Heine-Borel, and (5) every Cauchy sequence in $\mathbb{R}^k$ is convergent.

From Theorem 4.3.7, we know that for sequences in $\mathbb{R}^k$, being Cauchy is equivalent to being convergent. Since the equivalence can not be claimed over arbitrary metric spaces, the presence of that property receives a special designation.

Definition 4.3.9 *A metric space X is said to be* ***complete*** *if and only if for every sequence in X, the sequence being Cauchy is equivalent to it being convergent in X.*

Remark 4.3.10 *As noted earlier, $\mathbb{R}^k$ is complete. In view of Theorem 4.2.10, any compact metric space is complete. Finally because every closed subset of a metric space contains all of its limit points and the limit of a sequence is a limit point, we also have that every closed subset of a complete metric space is complete.*

It is always nice to find other conditions that ensure convergence of a sequence without actually having the find its limit. We know that compactness of the metric space allows us to deduce convergence from being Cauchy. On the other hand, we know that, in $\mathbb{R}^k$, compactness is equivalent to being closed and bounded. From the Bolzano-Weierstrass Theorem, boundedness of a sequence gives us a convergent subsequence. The sequence $\{i^n\}_{n=1}^{\infty}$ of elements in $\mathbb{C}$ quickly illustrates that boundedness of a sequence is not enough to give us convergence of the whole sequence. The good news is that, in $\mathbb{R}^1$, boundedness coupled with being increasing or decreasing will do the job.

Definition 4.3.11 *A sequence of real numbers* $\{x_n\}_{n=1}^{\infty}$ *is*

(a) ***monotonically increasing*** *if and only if* $(\forall n)\,(n \in \mathbb{J} \Rightarrow x_n \leq x_{n+1})$ *and*

(b) ***monotonically decreasing*** *if and only if* $(\forall n)\,(n \in \mathbb{J} \Rightarrow x_n \geq x_{n+1})$.

Definition 4.3.12 *The* ***class of monotonic sequences*** *consists of all the sequences in* $\mathbb{R}^1$ *that are either monotonically increasing or monotonically decreasing.*

Example 4.3.13 *For each* $n \in \mathbb{J}$, $\left(\dfrac{n+1}{n}\right)^n \geq 1 = \dfrac{(n+1)!}{(n+1)\,n!}$. *It follows that*

$$\frac{n!}{n^n} \geq \frac{(n+1)!}{(n+1)\,(n+1)^n}.$$

Consequently, $\left\{\dfrac{n!}{n^n}\right\}_{n=1}^{\infty}$ *is monotonically decreasing.*

Theorem 4.3.14 *Suppose that* $\{x_n\}_{n=1}^{\infty}$ *is monotonic. Then* $\{x_n\}_{n=1}^{\infty}$ *converges if and only if* $\{x_n\}_{n=1}^{\infty}$ *is bounded.*

Excursion 4.3.15 *Fill in what is missing in order to complete the following proof for the case when* $\{x_n\}_{n=1}^{\infty}$ *is monotonically decreasing.*

Proof. *By Lemma 4.1.10, if* $\{x_n\}_{n=1}^{\infty}$ *converges, then* $\underset{(1)}{\underline{\hspace{6cm}}}$.

Now suppose that $\{x_n\}_{n=1}^{\infty}$ *is monotonically decreasing and bounded. Let* $P = \{x_n : n \in \mathbb{J}\}$. *If* P *is finite, then there is at least one* k *such that* $x_k \in P$ *and, for infinitely many* $j \in \mathbb{J}$, *we have that* $x_j = x_k$. *On the other hand we have that* $x_{k+m} \geq x_{(k+m)+1}$ *for all* $m \in \mathbb{J}$. *It follows that* $\{x_n\}_{n=1}^{\infty}$ *is eventually a constant sequence which is convergent to* x_k. *If* P *is infinite and bounded, then from the* *greatest lower bound property of the reals, we can let* $g = \inf(P)$. *Because* g *is the greatest lower bound,*

$$(\forall n)\left(n \in \mathbb{J} \Rightarrow \underset{(2)}{\underline{\hspace{3cm}}}\right).$$

Suppose that $\varepsilon > 0$ *is given. Then there exists a positive integer* M *such that* $g \leq x_M < g + \varepsilon$; *otherwise,* $\underset{(3)}{\underline{\hspace{8cm}}}$. *Because* $\{x_n\}_{n=1}^{\infty}$ *is* $\underset{(4)}{\underline{\hspace{3cm}}}$, *the transitivity of less than or equal to yields that, for all* $n > M$, $g \leq x_n < g + \varepsilon$. *Hence,* $n > M \Rightarrow |x_n - g| < \varepsilon$. *Since* $\varepsilon > 0$ *was arbitrary, we conclude that* $\underset{(5)}{\underline{\hspace{5cm}}}$. ■

*******Acceptable responses are: (1) it is bounded, (2) $g \leq x_n$, (3) $g + \varepsilon$ would be a lower bound that is greater than g, (4) decreasing, and (5) $\lim_{n \to \infty} x_n = g$.*******

4.3.1 Upper and Lower Bounds

Our next definition expands the limit notation to describe sequences that are tending to infinity or negative infinity.

Definition 4.3.16 *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence of real numbers. Then*

(a) $x_n \to \infty$ *as* $n \to \infty$ *if and only if*

$$(\forall K)\left(K \in \mathbb{R}^1 \Rightarrow (\exists M)\,(M \in \mathbb{J} \wedge (\forall n)\,(n > M \Rightarrow x_n \geq K))\right)$$

and

(b) $x_n \to -\infty$ *as* $n \to \infty$ *if and only if*

$$(\forall K)\left(K \in \mathbb{R}^1 \Rightarrow (\exists M)\,(M \in \mathbb{J} \wedge (\forall n)\,(n > M \Rightarrow x_n \leq K))\right).$$

In the first case, we write $\lim_{n \to \infty} x_n = \infty$ *and in the second case we write* $\lim_{n \to \infty} x_n = -\infty$.

Definition 4.3.17 *For* $\{x_n\}_{n=1}^{\infty}$ *be a sequence of real numbers, let* E *denote the set of all subsequential limits in the extended real number system (this means that* ∞ *and/or* $-\infty$ *are included if needed). Then the* ***limit superior*** *of* $\{x_n\}_{n=1}^{\infty}$ *is* $x^* = \sup(E)$ *and the* ***limit inferior*** *of* $\{x_n\}_{n=1}^{\infty}$ *is* $x_* = \inf(E)$.

We will use $\limsup\limits_{n\to\infty} x_n$ to denote the limit superior and $\liminf\limits_{n\to\infty} x_n$ to denote the limit inferior of $\{x_n\}_{n=1}^{\infty}$.

Example 4.3.18 *For each* $n \in \mathbb{J}$, *let* $a_n = 1 + (-1)^n + \dfrac{1}{2^n}$. *Then the* $\limsup\limits_{n\to\infty} a_n = 2$ *and* $\liminf\limits_{n\to\infty} a_n = 0$.

Excursion 4.3.19 *Find the limit superior and the limit inferior for each of the following sequences.*

1. $\left\{s_n = \dfrac{n + (-1)^n (2n+1)}{n}\right\}_{n=1}^{\infty}$

2. $\left\{s_n = (-1)^{n+1} + \sin\dfrac{\pi n}{4}\right\}_{n=1}^{\infty}$

3. $\left\{s_n = \left(1 + \dfrac{1}{n}\right)\left(1 + \sin\dfrac{\pi n}{2}\right)\right\}_{n=1}^{\infty}$

4. $\left\{s_n = -\dfrac{n}{4} + \left\lceil \dfrac{n}{4} \right\rceil + (-1)^n\right\}_{n=1}^{\infty}$

For (1), we have two convergent subsequences to consider; $s_{2n} \to 3$ while $s_{2n-1} \to -1$ and you should have concluded that $\limsup_{n\to\infty} s_n = 3$ and $\liminf_{n\to\infty} s_n = -1$. In working on (2), you should have gotten 5 subsequential limits: $s_{4k} \to -1$, $\{s_{4k+1}\}$ and $\{s_{4k+3}\}$ give two subsequential limits, $1+\dfrac{\sqrt{2}}{2}$ for k even and $1-\dfrac{\sqrt{2}}{2}$ for k odd; $\{s_{4k+2}\}$ also gives two subsequential limits, -2 for k odd and 0 for k even. Comparison of the 5 subsequential limits leads to the conclusion that $\limsup_{n\to\infty} s_n = 1+\dfrac{\sqrt{2}}{2}$ and $\liminf_{n\to\infty} s_n = -2$. The sequence given in (3) leads to three subsequential limits, namely, $0, 1,$, and 2 which leads to the conclusion that $\limsup_{n\to\infty} s_n = 2$ and $\liminf_{n\to\infty} s_n = 0$. Finally, for (4), the subsequences $\{s_{4k}\}$, $\{s_{4k+1}\}$, $\{s_{4k+2}\}$, and $\{s_{4k+3}\}$ give limits of $1, -\dfrac{1}{4}, \dfrac{3}{2}$, and $-\dfrac{3}{4}$, respectively; hence, $\limsup_{n\to\infty} s_n = \dfrac{3}{2}$ and $\liminf_{n\to\infty} s_n = -\dfrac{3}{4}$.

Theorem 4.3.20 *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers and E be the set of (finite) subsequential limits of the sequence plus possibly $+\infty$ and $-\infty$. Then*

(a) $\limsup_{n\to\infty} s_n \in E$, *and*

(b) $(\forall x)\left(\left(x > \limsup_{n\to\infty} s_n\right) \Rightarrow (\exists M)\,(n > M \Rightarrow s_n < x)\right)$.

Moreover, $\limsup_{n\to\infty} s_n$ *is the only real number that has these two properties.*

Excursion 4.3.21 *Fill in what is missing in order to complete the following proof of the theorem.*

Proof. *For the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$, let E denote the set of subsequential limits of the sequence, adjoining $+\infty$ and/or $-\infty$ if needed, and $s^* = \limsup_{n\to\infty} s_n$.*

Proof of part (a): If $s^* = \infty$, then E is unbounded. Thus $\{s_n\}_{n=1}^{\infty}$ is not bounded above and we conclude that there is a subsequence $\left\{s_{n_k}\right\}_{k=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} s_{n_k} = \infty$.

If $s^* = -\infty$, then $\{s_n\}_{n=1}^{\infty}$ has no finite subsequential limits; i.e., $-\infty$ is the only element of E. It follows that $\lim_{n\to\infty} s_n = -\infty$.

Suppose that $s^* \in \mathbb{R}$. Then E is bounded above and contains at least one element. By CN Theorem 4.1.20, the set E is $\underset{(1)}{\underline{\hspace{3cm}}}$. It follows from CN Theorem $\underset{(2)}{\underline{\hspace{3cm}}}$ that $s^* = \sup(E) \in \overline{E} = E$.

Proof of part (b): Suppose that there exists $x \in \mathbb{R}$ such that $x > s^*$ and $s_n \geq x$ for infinitely many natural numbers n. Then there exists a subsequence of $\{s_n\}_{n=1}^{\infty}$ that converges to some real number y such that $\underset{(3)}{\underline{\hspace{3cm}}}$. From the triangular inequality, $y > s^*$. But $y \in E$ and $y > s^*$ contradicts the definition of $\underset{(4)}{\underline{\hspace{3cm}}}$. It follows that, for any $x > s^*$ there are at most finitely many $n \in \mathbb{J}$ for which $\underset{(5)}{\underline{\hspace{3cm}}}$. Hence, for any $x > s^*$ there exists a positive integer M such that $n > M$ implies that $s_n < x$.

Proof of uniqueness. Suppose that p and q are distinct real numbers that satisfy property (b). Then

$$(\forall x)\left((x > p) \Rightarrow (\exists M)\left(n > M \Rightarrow s_n < x\right)\right)$$

and

$$(\forall x)\left((x > q) \Rightarrow (\exists K)\left(n > K \Rightarrow s_n < x\right)\right).$$

Without loss of generality we can assume that $p < q$. Then there exists $w \in \mathbb{R}$ such that $p < w < q$. Since $w > p$ there exists $M \in \mathbb{J}$ such that $n > M$ implies that $s_n < w$. In particular, at most finitely many of the s_k satisfy $\underset{(6)}{\underline{\hspace{3cm}}}$. Therefore, q cannot be the limit of any subsequence of $\{s_n\}_{n=1}^{\infty}$ from which it follows that $q \notin E$; i.e., q does not satisfy property (a). ■

Acceptable responses are: (1) closed, (2) 3.3.26, (3) $y \geq x$, (4) $\sup E$, (5) $s_n \geq x$, and (6) $q > s_k > w$.

Remark 4.3.22 *Note that, if* $\{s_n\}_{n=1}^{\infty}$ *is a convergent sequence of real numbers, say* $\lim s_n = s_0$, *then the set of subsequential limits is just* $\{s_0\}$ *and it follows that*

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n.$$

Theorem 4.3.23 *If* $\{s_n\}_{n=1}^{\infty}$ *and* $\{t_n\}_{n=1}^{\infty}$ *are sequences of real numbers and there exists a positive integer* M *such that* $n > M$ *implies that* $s_n \leq t_n$, *then*

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n \quad and \quad \limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n.$$

Excursion 4.3.24 *Offer a well presented justification for Theorem 4.3.23.*

4.4 Some Special Sequences

This section offers some limits for sequences with which you should become familiar. Space is provided so that you can fill in the proofs. If you get stuck, proofs can be found on page 58 of our text.

Lemma 4.4.1 *For any fixed positive real number,* $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0.$

Proof. For $\varepsilon > 0$, let $M = M(\varepsilon) = \left\lceil \left(\frac{1}{\varepsilon}\right)^{1/p} \right\rceil$.

■

Lemma 4.4.2 *For any fixed complex number x such that $|x| < 1$, $\lim_{n\to\infty} x^n = 0$.*

Proof. If $x = 0$, then $x^n = 0$ for each $n \in \mathbb{J}$ and $\lim_{n\to\infty} x^n = 0$. Suppose that x is a fixed complex number such that $0 < |x| < 1$. For $\varepsilon > 0$, let

$$M = M(\varepsilon) = \begin{cases} 1 & , \quad \text{for } \varepsilon \geq 1 \\ \left\lceil \dfrac{\ln(\varepsilon)}{\ln|x|} \right\rceil & , \quad \text{for } \varepsilon < 1 \end{cases} .$$

■

The following theorem makes use of the Squeeze Principle and the Binomial Theorem. The special case of the latter that we will use is that, for $n \in \mathbb{J}$ and

$\zeta \in \mathbb{R} - \{-1\}$,

$$(1+\zeta)^n = \sum_{k=0}^{n} \binom{n}{k} \zeta^k, \text{ where } \binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

In particular, if $\zeta > 0$ we have that $(1+\zeta)^n \geq 1 + n\zeta$ and $(1+\zeta)^n > \binom{n}{k}\zeta^k$ for each k, $1 \leq k \leq n$.

Theorem 4.4.3 *(a) If* $p > 0$, *then* $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$.

(b) We have that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

(c) If $p > 0$ *and* $\alpha \in \mathbb{R}$, *then* $\lim\limits_{n\to\infty} \dfrac{n^\alpha}{(1+p)^n} = 0$.

Proof of (a). We need prove the statement only for the case of $p > 1$; the result for $0 < p < 1$ will follow by substituting $\dfrac{1}{p}$ in the proof of the other case. If $p > 1$, then set $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$ and from the Binomial Theorem,

$$1 + nx_n \leq (1 + x_n)^n = p$$

and

$$0 < x_n \leq \frac{p-1}{n}.$$

■

Proof of (b). Let $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$ and, from the Binomial Theorem,

$$n = (1 + x_n)^n \geq \frac{n\,(n-1)}{2} x_n^2.$$

■

Proof of (c). Let k be a positive integer such that $k > \alpha$. For $n > 2k$,

$$(1+p)^n > \binom{n}{k} p^k = \frac{n\,(n-1)\,(n-1)\cdots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}$$

and

$$0 < \frac{n^\alpha}{(1+p)^n} < \underline{\hspace{6cm}}.$$

■

4.5 Series of Complex Numbers

For our discussion of series, we will make a slight shift is subscripting; namely, it will turn out to be more convenient for us to have our initial subscript be 0 instead of 1. Given any sequence of complex numbers $\{a_k\}_{k=0}^{\infty}$, we can associate (or derive) a related sequence $\{S_n\}_{n=0}^{\infty}$ where $S_n = \sum_{k=0}^{n} a_k$ called the sequence of nth partial sums. The associated sequence allows us to give precise mathematical meaning to the idea of “finding an infinite sum.”

Definition 4.5.1 *Given a sequence of complex numbers $\{a_k\}_{k=0}^{\infty}$, the symbol $\sum_{k=0}^{\infty} a_k$ is called an* ***infinite series*** *or simply a* ***series****. The symbol is intended to suggest an infinite summation*

$$a_0 + a_1 + a_2 + a_3 + \cdots$$

and each a_n is called a ***term*** *in the series. For each $n \in \mathbb{J} \cup \{0\}$, let $S_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n$. Then $\{S_n\}_{n=0}^{\infty}$ is called the* ***sequence of*** *nth* ***partial sums*** *for $\sum_{k=0}^{\infty} a_k$.*

On the surface, the idea of adding an infinite number of numbers has no real meaning which is why the series has been defined just as a symbol. We use the associated sequence of nth partial sums to create an interpretation for the symbol that is tied to a mathematical operation that is well defined.

Definition 4.5.2 *An infinite series $\sum_{k=0}^{\infty} a_k$ is said to be* ***convergent*** *to the complex number S if and only if the sequence of nth partial sums $\{S_n\}_{n=0}^{\infty}$ is convergent to S; when this occurs, we write $\sum_{k=0}^{\infty} a_k = S$. If $\{S_n\}_{n=0}^{\infty}$ does not converge, we say that the series is* ***divergent****.*

Remark 4.5.3 *The way that convergence of series is defined, makes it clear that we really aren't being given a brand new concept. In fact, given any sequence $\{S_n\}_{n=0}^{\infty}$, there exists a sequence $\{a_k\}_{k=0}^{\infty}$ such that $S_n = \sum_{k=1}^{n} a_k$ for every $k \in \mathbb{J} \cup \{0\}$: To see this, simply choose $a_0 = S_0$ and $a_k = S_k - S_{k-1}$ for $k \geq 1$. We will treat sequences and series as separate ideas because it is convenient and useful to do so.*

The remark leads us immediately to the observation that for a series to converge it is necessary that the terms go to zero.

Lemma 4.5.4 (kth term test) *If the series $\sum_{k=0}^{\infty} a_k$ converges, then $\lim\limits_{k\to\infty} a_k = 0$.*

Proof. Suppose that $\sum_{k=0}^{\infty} a_k = S$. Then $\lim\limits_{k\to\infty} S_k = S$ and $\lim\limits_{k\to\infty} S_{k-1} = S$. Hence, by Theorem 4.3.2(a),

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = S - S = 0.$$

■

Remark 4.5.5 *To see that the converse is not true, note that the harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

is divergent which is a consequence of the following excursion.

Excursion 4.5.6 *Use the Principle of Mathematical Induction to prove that, for* $\sum_{k=1}^{\infty} \frac{1}{k}$, $S_{2^n} > 1 + \dfrac{n}{2}$.

Excursion 4.5.7 *Use the definition of convergence (divergence) to discuss the following series.*

(a) $\sum_{k=1}^{\infty} \sin \frac{\pi k}{4}$

(b) $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

The first example can be claimed as divergent by inspection, because the nth term does not go to zero. The key to proving that the second one converges is noticing that $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$; in fact, the given problem is an example of what is known as a telescoping sum.

The following set of lemmas are just reformulations of results that we proved for sequences.

Lemma 4.5.8 (Cauchy Criteria for Series Convergence) *The series (of complex numbers)* $\sum_{k=0}^{\infty} a_k$ *is convergent if and only if for every* $\varepsilon > 0$ *there exists a positive integer* $M = M(\varepsilon)$ *such that* $(\forall m)(\forall n)(m, n > M \Rightarrow |S_m - S_n| < \epsilon)$.

Proof. The lemma holds because the complex sequence of nth partial sums $\{S_n\}_{n=0}^{\infty}$ is convergent if and only if it is Cauchy. This equivalence follows from the combination of Theorem 4.2.9 and Theorem 4.3.6(b). ■

Remark 4.5.9 *We will frequently make use of the following alternative formulation for the sequence of* nth *partial sums being Cauchy. Namely,* $\{S_n\}_{n=0}^{\infty}$ *is Cauchy if and only if for every* $\epsilon > 0$, *there exists a positive integer* M *such that* $n > M$ *implies that* $|S_{n+p} - S_n| = \left|\sum_{k=n+1}^{n+p} a_k\right| < \epsilon$, *for* $p = 1, 2, ...$

Lemma 4.5.10 *For the series (of complex numbers)* $\sum_{k=0}^{\infty} a_k$, *let* $\operatorname{Re} a_k = x_k$ *and* $\operatorname{Im} a_k = y_k$. *Then* $\sum_{k=0}^{\infty} a_k$ *is convergent if and only if* $\sum_{k=0}^{\infty} x_k$ *and* $\sum_{k=0}^{\infty} y_k$ *are convergent (real) sequences.*

Proof. For the complex series $\sum_{k=0}^{\infty} a_k$,

$$S_n = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} x_k + i \sum_{k=0}^{n} y_k = \left(\sum_{k=0}^{n} x_k, \sum_{k=0}^{n} y_k \right).$$

Consequently, the result is simply a statement of Lemma 4.3.1 for the case $n = 2$. ■

Lemma 4.5.11 *Suppose that* $\sum_{k=0}^{\infty} a_k$ *is a series of nonnegative real numbers. Then* $\sum_{k=0}^{\infty} a_k$ *is convergent if and only if its sequence of* nth *partial sums is bounded.*

Proof. Suppose that $\sum_{k=0}^{\infty} a_k$ is a series of nonnegative real numbers. Then $\{S_n\}_{n=0}^{\infty}$ is a monotonically increasing sequence. Consequently, the result follows from Theorem 4.3.14. ■

Lemma 4.5.12 *Suppose that* $\sum_{k=0}^{\infty} u_k$ *and* $\sum_{k=0}^{\infty} v_k$ *are convergent to* U *and* V, *respectively., and* c *is a nonzero constant. Then*

1. *$\sum_{k=0}^{\infty}(u_k \pm v_k) = U \pm V$ and*
2. *$\sum_{k=0}^{\infty} cu_k = cU$.*

Most of our preliminary discussion of series will be with series for which the terms are positive real numbers. When not all of the terms are positive reals, we first check for absolute convergence.

Definition 4.5.13 *The series $\sum_{j=0}^{\infty} a_j$ is said to be **absolutely convergent** if and only if $\sum_{j=0}^{\infty} |a_j|$ converges. If $\sum_{j=0}^{\infty} a_j$ converges and $\sum_{j=0}^{\infty} |a_j|$ diverges, then the series $\sum_{j=0}^{\infty} a_j$ is said to be **conditionally convergent**.*

After the discussion of some tests for absolute convergence, we will see that absolute convergence implies convergence. Also, we will justify that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

4.5.1 Some (Absolute) Convergence Tests

While the definition may be fun to use, we would like other means to determine convergence or divergence of a given series. This leads us to a list of tests, only a few of which are discussed in this section.

Theorem 4.5.14 (Comparison Test) *Suppose that $\sum_{k=0}^{\infty} a_k$ is a series (of complex numbers).*

(a) If there exists a positive integer M such that $(\forall k)\,(k \geq M \Rightarrow |a_k| \leq c_k)$ for real constants c_k and $\sum_{k=0}^{\infty} c_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges absolutely.

(b) If there exists a positive integer M such that $(\forall k)\,(k \geq M \Rightarrow |a_k| \geq d_k \geq 0)$ for real constants d_k and $\sum_{k=0}^{\infty} d_k$ diverges, then $\sum_{k=0}^{\infty} |a_k|$ diverges.

Proof of (a). Suppose that $\sum_{k=0}^{\infty} a_k$ is a series (of complex numbers), there exists a positive integer M such that $(\forall k)\,(k \geq M \Rightarrow |a_k| \leq c_k)$, and $\sum_{k=0}^{\infty} c_k$ converges. For fixed $\varepsilon > 0$, there exists a positive integer K such that $n > K$ and $p \in \mathbb{J}$ implies that

$$\left|\sum_{k=n+1}^{n+p} c_k\right| = \sum_{k=n+1}^{n+p} c_k < \epsilon.$$

For $n > M^* = \max\{M, K\}$ and any $p \in \mathbb{J}$, it follows from the triangular inequality that

$$\left|\sum_{k=n+1}^{n+p} a_k\right| \le \sum_{k=n+1}^{n+p} |a_k| \le \sum_{k=n+1}^{n+p} c_k < \epsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\sum_{k=0}^{\infty} a_k$ converges. ■

Proof of (b). Suppose that $\sum_{k=0}^{\infty} a_k$ is a series (of real numbers), there exists a positive integer M such that $(\forall k)\,(k \ge M \Rightarrow |a_k| \ge d_k \ge 0)$, and $\sum_{k=0}^{\infty} d_k$ diverges. From Lemma 4.5.11, $\left\{\sum_{k=0}^{n} d_k\right\}_{n=0}^{\infty}$ is an unbounded sequence. Since

$$\sum_{k=M}^{n} |a_k| \ge \sum_{k=M}^{n} d_k$$

for each $n > M$, it follows that $\left\{\sum_{k=0}^{n} |a_k|\right\}_{n=0}^{\infty}$ is an unbounded. Therefore, $\sum_{k=0}^{\infty} |a_k|$ diverges. ■

In order for the Comparison Tests to be useful, we need some series about which convergence or divergence behavior is known. The best known (or most famous) series is the Geometric Series.

Definition 4.5.15 *For a nonzero constant a, the series $\sum_{k=0}^{\infty} ar^k$ is called a* ***geometric series****. The number r is the* ***common ratio****.*

Theorem 4.5.16 (Convergence Properties of the Geometric Series) *For $a \neq 0$, the geometric series $\sum_{k=0}^{\infty} ar^k$ converges to the sum $\dfrac{a}{1-r}$ whenever $0 < |r| < 1$ and diverges whenever $|r| \ge 1$.*

Proof. The claim will follow upon showing that, for each $n \in \mathbb{J} \cup \{0\}$,

$$\sum_{k=0}^{n} ar^k = \frac{a\left(1 - r^{n+1}\right)}{1 - r}.$$

■

The proof of the next result makes use of the "regrouping" process that was applied to our study of the harmonic series.

Theorem 4.5.17 *If* $\left\{a_j\right\}_{j=0}^{\infty}$ *is a monotonically decreasing sequence of nonnegative real numbers, then the series* $\sum_{j=0}^{\infty} a_j$ *is convergent if and only if* $\sum_{j=0}^{\infty} 2^j a_{2^j}$ *converges.*

Excursion 4.5.18 *Fill in the two blanks in order to complete the following proof of Theorem 4.5.17.*

Proof. Suppose that $\left\{a_j\right\}_{j=0}^{\infty}$ is a monotonically decreasing sequence of nonnegative real numbers. For each $n, k \in \mathbb{J} \cup \{0\}$, let

$$S_n = \sum_{j=0}^{n} a_j \quad and \quad T_k = \sum_{j=0}^{k} 2^j a_{2^j}.$$

Note that, because $\left\{a_j\right\}_{j=0}^{\infty}$ is a monotonically decreasing sequence, for any $j \in \mathbb{J} \cup \{0\}$ and $m \in \mathbb{J}$,

$$(m+1)\, a_j \geq a_j + a_{j+1} + \cdots + a_{j+m} \geq (m+1)\, a_{j+m},$$

while $\left\{a_j\right\}_{j=0}^{\infty}$ a sequence of nonnegative real numbers yields that $\{S_n\}$ and $\{T_k\}$ are monotonically decreasing sequences. For $n < 2^k$,

$$S_n \leq a_0 + a_1 + \underbrace{(a_2 + a_3)}_{2^1\ terms} + \underbrace{(a_4 + a_5 + a_6 + a_7)}_{2^2\ terms} + \cdots + \underbrace{\left(a_{2^k} + \cdots + a_{2^{k+1}-1}\right)}_{2^k\ terms}$$

from which it follows that

(1)

For $n > 2^k$,

$$S_n \geq a_0 + a_1 + a_2 + \underbrace{(a_3 + a_4)}_{2^1\ terms} + \underbrace{(a_5 + a_6 + a_7 + a_8)}_{2^2\ terms} \cdots + \underbrace{\left(a_{2^{k-1}+1} + \cdots + a_{2^k}\right)}_{2^{k-1}\ terms}$$

from which it follows that

(2)

The result now follows because we have that $\{S_n\}$ and $\{T_k\}$ are simultaneously bounded or unbounded. ■

For (1), the grouping indicated leads to $S_n \leq a_1 + a_0 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = a_1 + T_k$, while the second regrouping yields that $S_n \geq a_0 + a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k} = \frac{1}{2} a_0 + a_1 + \frac{1}{2} T_k$.

As an immediate application of this theorem, we obtain a family of real series for which convergence and divergence can be claimed by inspection.

Theorem 4.5.19 (Convergence Properties of p-series) *The series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges whenever* $p > 1$ *and diverges whenever* $p \leq 1$.

Proof. If $p \leq 0$, the $p-$series diverges by the kth term test. If $p > 0$, then $\left\{ a_n = \frac{1}{n^p} \right\}_{n=1}^{\infty}$ is a monotonically decreasing sequence of nonnegative real numbers. Note that

$$\sum_{j=0}^{\infty} 2^j a_{2^j} = \sum_{j=0}^{\infty} 2^j \frac{1}{\left(2^j\right)^p} = \sum_{j=0}^{\infty} \left(2^{(1-p)}\right)^j .$$

Now use your knowledge of the geometric series to finish the discussion.

■

A similar argument yields the following result with is offered without proof. It is discussed on page 63 of our text.

Lemma 4.5.20 *The series* $\sum_{j=2}^{\infty} \dfrac{1}{j\,(\ln j)^p}$ *converges whenever* $p > 1$ *and diverges whenever* $p \leq 1$.

Excursion 4.5.21 *Discuss the convergence (or divergence) of each of the following.*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n^2+3n-1}$

Notice that all of the series given in this excursion are over the positive reals; thus, checking for absolute convergence is the same as checking for convergence. At this point, we only the n^{th} term test, Comparison, recognition as a p-series, or rearrangement in order to identify the given as a geometric series. For (a), noticing that, for each $n \in \mathbb{J}$, $\frac{n}{n^2+1} \geq \frac{n}{n^2+n} = \frac{1}{n+1}$ allows us to claim divergence by comparison with the "shifted" harmonic series. The series given in (b) is convergent as a p-series for $p = 3$. Because $\lim \frac{n-1}{2n+1} = \frac{1}{2} \neq 0$ the series given in (c) diverges by the n^{th} term test. Finally, since $3n - 1 > 0$ for each $n \in \mathbb{J}$, $\frac{3}{n^2+3n-1} \leq \frac{3}{n^2}$ which allows us to claim convergence of the series given in (d) by comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n^2}$ which is convergent as a constant multiple times the p-series with $p = 2$.

When trying to make use of the Comparison Test, it is a frequent occurrence that we know the nature of the series with which to make to comparison almost by inspection though the exact form of a beneficial comparison series requires some creative algebraic manipulation. In the last excursion, part (a) was a mild example of this phenomenon. A quick comparison of the degrees of the rational functions that form the term suggest divergence by association with the harmonic series, but when we see that $\frac{n}{n^2+1} \not\geq \frac{1}{n}$ we have to find some way to manipulate the expression $\frac{n}{n^2+1}$ more creatively. I chose to illustrate the "throwing more in the denominator" argument; as an alternative, note that for any natural number n, $n^2 \geq 1 \Rightarrow 2n^2 \geq n^2 + 1 \Rightarrow \frac{n}{n^2+1} \geq \frac{1}{2n}$ which would have justified divergence by comparison with a constant multiple of the harmonic series. We have a nice variation of the comparison test that can enable us to bypass the need for the algebraic manipulations. We state here and leave its proof as exercise.

Theorem 4.5.22 (Limit Comparison Test) *Suppose that* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *are* *such that* $a_n \geq 0$, $b_n \geq 0$ *for each* $n \in \mathbb{J} \cup \{0\}$, *and* $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$. *Then either* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *both converge or both diverge.*

We have two more important and well known tests to consider at this point.

Theorem 4.5.23 (Ratio Test) *The series* $\sum_{k=0}^{\infty} a_k$

(i) converges absolutely if $\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$*;*

(ii) diverges if there exists a nonnegative integer M *such that* $k > M$ *implies that* $\left|\frac{a_{k+1}}{a_k}\right| \geq 1$.

Proof. Suppose that the series $\sum_{k=0}^{\infty} a_k$ is such that $\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$. It follows that we can find a positive real number β such that $\beta < 1$ and there exists an $M \in \mathbb{J}$ such that $n > M$ implies that $\left|\frac{a_{k+1}}{a_k}\right| < \beta$. It can be shown by induction that, for each $p \in \mathbb{J}$ and $n > M$, $\left|a_{n+p}\right| < \beta^p \left|a_n\right|$. In particular, for $n \geq M+1$ and $p \in \mathbb{J} \cup \{0\}$, $\left|a_{n+p}\right| < \beta^p \left|a_{M+1}\right|$. Now, the series $\sum_{p=1}^{\infty} \left|a_{M+1}\right| \beta^p$ is convergent as a geometric series with ratio less than one. Hence, $\sum_{j=M+1}^{\infty} a_j = \sum_{p=1}^{\infty} a_{M+p}$ is absolutely convergent by comparison from which it follows that $\sum_{k=0}^{\infty} a_k$ is absolutely convergent.

Suppose that the series $\sum_{k=0}^{\infty} a_k$ is such that there exists a nonnegative integer M for which $k > M$ implies that $\left|\frac{a_{k+1}}{a_k}\right| \geq 1$. Briefly justify that this yields divergence as a consequence of the n^{th} term test.

■

Remark 4.5.24 *Note that* $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$ *leads to no conclusive information concerning the convergence or divergence of* $\sum_{k=0}^{\infty} a_k$.

Example 4.5.25 *Use the Ratio Test to discuss the convergence of each of the following:*

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n-1)!}$

   For $a_n = \frac{1}{(n-1)!}$, $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{1}{n!}(n-1)!\right| = \frac{1}{n} \to 0$ *as* $n \to \infty$. *Hence,* $\limsup\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$ *and we conclude that the series is (absolutely) convergent from the ratio test.*

2. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n}$

   Let $a_n = \frac{n^2}{2^n}$. *Then* $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}\right| = \frac{1}{2}\left(1+\frac{1}{n}\right)^2 \to \frac{1}{2}$ *as* $n \to \infty$. *Thus,* $\limsup\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{2} < 1$ *and we conclude that the given series is (absolutely) convergent.*

Theorem 4.5.26 (Root Test) *For* $\sum_{k=0}^{\infty} a_k$, *let* $\alpha = \limsup\limits_{k\to\infty}\sqrt[k]{|a_k|}$,

(i) if $0 \le \alpha < 1$, *then* $\sum_{k=0}^{\infty} a_k$ *converges absolutely;*

(ii) if $\alpha > 1$, *then* $\sum_{k=0}^{\infty} a_k$ *diverges; and*

(iii) if $\alpha = 1$, *then no information concerning the convergence or divergence of* $\sum_{k=0}^{\infty} a_k$ *can be claimed.*

Proof. For $\sum_{k=0}^{\infty} a_k$, let $\alpha = \limsup\limits_{k\to\infty}\sqrt[k]{|a_k|}$. If $\alpha < 1$, there exists a real number β such that $\alpha < \beta < 1$. Because α is a supremum of subsequential limits and $\alpha < \beta < 1$, by Theorem 4.3.20, there exists a positive integer M such that $n > M$ implies that $\sqrt[n]{|a_n|} < \beta$; i.e., $|a_n| < \beta^n$ for all $n > M$. Since $\displaystyle\sum_{j=M+1}^{\infty}\beta^j$ is convergent as a geometric series (that sums to $\frac{\beta^{m+1}}{1-\beta}$), we conclude that $\sum_{k=0}^{\infty} |a_k|$ converges; that is, $\sum_{k=0}^{\infty} a_k$ converges absolutely.

Briefly justify that $\alpha > 1$ leads to divergence of $\sum_{k=0}^{\infty} a_k$ as a consequence of the n^{th} term test.

Finally, since $\alpha = \limsup_{k\to\infty} \sqrt[k]{|a_k|} = 1$ for the p-series, we see that no conclusion can be drawn concerning the convergence of divergence of the given series.

■

Example 4.5.27 *Use the Root Test, to establish the convergence of* $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$.

From Theorem 4.4.3(a) and (b), $\lim_{n\to\infty} \sqrt[n]{2n} = 1$. *Hence,*

$$\limsup_{k\to\infty} \sqrt[k]{|\frac{k}{2^{k-1}}|} = \lim_{k\to\infty} \sqrt[k]{2\left(\frac{k}{2^k}\right)} = \lim_{k\to\infty} \frac{\sqrt[k]{2k}}{2} = \frac{1}{2} < 1$$

from which we claim (absolute) convergence of the given series.

Thus far our examples of applications of the Ratio and Root test have led us to exam sequences for which $\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$ or $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{|a_k|}$. This relates back to the form of the tests that you should have seen with your first exposure to series tests, probably in frosh (or AP) calculus. Of course, the point of offering the more general statements of the tests is to allow us to study the absolute convergence of series for which appeal to the limit superior is necessary. The next two excursion are in the vein; the parts that are described seek to help you to develop more comfort with the objects that are examined in order to make use of the Ratio and Root tests.

Excursion 4.5.28 *For* $n \in \mathbb{J} \cup \{0\}$, *let* $a_j = \begin{cases} \left(\frac{1+i}{2}\right)^j & , \quad if\, 2 \mid j \\ \left(\frac{2}{5}\right)^j & , \quad if\, 2 \nmid j \end{cases}$.

1. *Find the first four terms of* $\left\{ \left| \frac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$.

2. *Find the first four terms of* $\left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty}$.

3. *Find* E_1 *the set of subsequential limits of* $\left\{ \left| \frac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$

4. *Find* E_2 *the set of subsequential limits of* $\left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty}$

5. *Find each of the following:*

   (a) $\limsup_{j \to \infty} \left\{ \left| \frac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$

(b) $\liminf\limits_{j\to\infty} \left\{\left|\frac{a_{j+1}}{a_j}\right|\right\}_{j=0}^{\infty}$

(c) $\limsup\limits_{j\to\infty} \left\{\sqrt[j]{|a_j|}\right\}_{j=1}^{\infty}$

(d) $\liminf\limits_{j\to\infty} \left\{\sqrt[j]{|a_j|}\right\}_{j=1}^{\infty}$

6. *Discuss the convergence of* $\sum\limits_{j=0}^{\infty} a_j$

*******For (1), we are looking at $\left\{\frac{2}{5}, \frac{5}{4}, \frac{16}{125}, \frac{125}{32}, \cdots\right\}$ while (2) is $\left\{\frac{2}{5}, \frac{\sqrt{2}}{2}, \frac{2}{5}, \frac{\sqrt{2}}{2}, \cdots\right\}$; for (3), if $c_j = \left|\frac{a_{j+1}}{a_j}\right|$, then the possible subsequential limits are given by looking at $\{c_{2j}\}$ and $\{c_{2j-1}\}$ and $E_1 = \{0, \infty\}$; if in (4) we let $d_j = \sqrt[j]{|a_j|}$, then consideration of $\{d_{2j}\}$ and $\{d_{2j-1}\}$ leads to $E_2 = \left\{\frac{\sqrt{2}}{2}, \frac{2}{5}\right\}$; For (3) and (4), we conclude that the requested values are ∞, 0, $\frac{\sqrt{2}}{2}$, and $\frac{2}{5}$, respectively. For the discussion of (6), note that The Ratio Test yields no information because neither (a) nor (b) is satisfied; in the other hand, from (5c), we see that $\limsup\limits_{j\to\infty} \left\{\sqrt[j]{|a_j|}\right\}_{j=1}^{\infty} = \frac{\sqrt{2}}{2} < 1$, from which we conclude that the given series is absolutely convergent. (As an aside, examination of $\{S_{2n}\}$ and $\{S_{2n-1}\}$ corresponding to $\sum_{j=0}^{\infty} a_j$ even allows us to conclude that the sum of the given series is $\frac{4}{1+2i} + \frac{10}{21} = \frac{134-168i}{105}$.)***

Excursion 4.5.29 *For $n \in \mathbb{J} \cup \{0\}$, let* $a_j = \begin{cases} \left(\frac{2}{3}\right)^{j+1} & , \quad if\, 2 \mid j \\ \left(\frac{2}{3}\right)^{j-1} & , \quad if\, 2 \nmid j \end{cases}$.

1. *Find the first four terms of* $\left\{ \left| \frac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$.

2. *Find the first four terms of* $\left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty}$.

3. *Find E_1 the set of subsequential limits of* $\left\{ \left| \frac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$

4. *Find E_2 the set of subsequential limits of* $\left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty}$

5. *Find each of the following:*

(a) $\limsup_{j\to\infty} \left\{ \left| \frac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$

(b) $\liminf_{j\to\infty} \left\{ \left| \frac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$

(c) $\limsup_{j\to\infty} \left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty}$

(d) $\liminf_{j\to\infty} \left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty}$

6. *Discuss the convergence of* $\sum_{j=1}^{\infty} a_j$

*******Response this time are: (1) $\left\{ \frac{3}{2}, \left(\frac{2}{3}\right)^3, \frac{3}{2}, \left(\frac{2}{3}\right)^3, \cdots \right\}$, (2) $\left\{ 1, \left(\frac{2}{3}\right)^{3/2} = \frac{2}{3}\sqrt{\frac{2}{3}}, \left(\frac{2}{3}\right)^{2/3} = \sqrt[3]{\frac{4}{9}}, \left(\frac{2}{3}\right)^{5/4} = \frac{4}{9}\sqrt[4]{\frac{2}{3}}, \cdots \right\}$; (3) $E_1 = \left\{ \frac{3}{2}, \frac{4}{9} \right\}$, (4) $E_2 = \left\{ \frac{2}{3} \right\}$ where this comes from separate consideration of $\lim_{j\to\infty} \left\{ \sqrt[2j]{|a_{2j}|} \right\}$ and $\lim_{j\to\infty} \left\{ \sqrt[2j-1]{|a_{2j-1}|} \right\}$, (5) the values are $\frac{3}{2}, \frac{4}{9}, \frac{2}{3}$ and $\frac{2}{3}$, respectively. Finally, the Ratio Test fails to offer information concerning convergence, however, the Root

Test yields that $\sum_{j=1}^{\infty} a_j$ is absolutely convergent. (Again, if we choose to go back to the definition, examination of the nth partial sums allows us to conclude that the series converges to 3.)*******

Remark 4.5.30 *Note that, if* $\limsup_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| > 1$ *for a series* $\sum_{k=0}^{\infty} a_k$*, the ratio test yields no information concerning the convergence of the series.*

4.5.2 Absolute Convergence and Cauchy Products

When the terms in the generating sequence for a series are not all nonnegative reals, we pursue the possibility of different forms of convergence.

The next result tells us that absolute convergence is a stronger condition than convergence

Lemma 4.5.31 *If* $\left\{a_j\right\}_{j=1}^{\infty}$ *is a sequence of complex numbers and* $\sum_{j=1}^{\infty} |a_j|$ *converges, then* $\sum_{j=0}^{\infty} a_j$ *converges and* $\left|\sum_{j=0}^{\infty} a_j\right| \leq \sum_{j=0}^{\infty} |a_j|$.

Proof. (if we were to restrict ourselves to real series) The following argument that is a very slight variation of the one offered by the author of our text applies only to series over the reals; it is followed by a general argument that applies to series of complex terms. Suppose $\left\{a_j\right\}_{j=1}^{\infty}$ is a sequence of real numbers such that $\sum_{j=1}^{\infty} |a_j|$ converges and define

$$v_j = \frac{|a_j| + a_j}{2} \text{ and } w_j = \frac{|a_j| - a_j}{2}.$$

Then $v_j - w_j = a_j$ while $v_j + w_j = |a_j|$. Furthermore,

$$a_j \geq 0 \text{ implies that } \quad v_j = a_j = |a_j| \text{ and } \quad w_j = 0$$

while

$$a_j < 0 \text{ implies that } \quad v_j = 0 \text{ and } \quad w_j = -a_j = |a_j|.$$

Consequently, $0 \leq v_n \leq |a_n|$ and $0 \leq u_n \leq |a_n|$ and, from the Comparison Test, it follows that $\sum_{j=0}^{\infty} v_j$ and $\sum_{j=0}^{\infty} w_j$ converge. By Lemma 4.5.12, $\sum_{j=0}^{\infty} \left(v_j - w_j\right)$ converges. Finally, since

$$-\left(v_j + w_j\right) \leq \left(v_j - w_j\right) \leq \left(v_j + w_j\right),$$

we see that

$$-\sum_{j=1}^{\infty}\left(v_j+w_j\right)\leq\sum_{j=1}^{\infty}\left(v_j-w_j\right)\leq\sum_{j=1}^{\infty}\left(v_j+w_j\right);$$

i.e., $-\sum_{j=1}^{\infty}\left|a_j\right|\leq\sum_{j=1}^{\infty}a_j\leq\sum_{j=1}^{\infty}\left|a_j\right|$. Hence $0\leq\left|\sum_{j=1}^{\infty}a_j\right|\leq\sum_{j=1}^{\infty}\left|a_j\right|$ ■

The following proof of the lemma in general makes use of the Cauchy Criteria for Convergence.

Proof. Suppose that $\left\{a_j\right\}_{j=1}^{\infty}$ is a sequence of complex numbers such that $\sum_{j=1}^{\infty}|a_j|$ converges and $\varepsilon>0$ is given. Then there exists a positive integer $M=M(\varepsilon)$ such that $(\forall m)(\forall n)(m,n>M\Rightarrow|S_m-S_n|<\epsilon)$ where $S_m=\sum_{j=1}^{m}|a_j|$. In particular, for any $p\in\mathbb{J}$ and $n>M$, $\sum_{j=n+1}^{n+p}|a_j|=\left|\sum_{j=n+1}^{n+p}|a_j|\right|<\varepsilon$. From the triangular inequality, it follows that $\left|\sum_{j=n+1}^{n+p}a_j\right|\leq\sum_{j=n+1}^{n+p}|a_j|<\varepsilon$ for any $p\in\mathbb{J}$ and $n>M$. Since $\varepsilon>0$ was arbitrary, we conclude that $\sum_{j=1}^{\infty}a_j$ converges by the Cauchy Criteria for Convergence. ■

Remark 4.5.32 *A re-read of the comparison, root and ratio tests reveals that they are actually tests for absolute convergence.*

Absolute convergence offers the advantage of allowing us to treat the absolutely convergence series much as we do finite sums. We have already discussed the term by term sums and multiplying by a constant. There are two kinds of product that come to mind: The first is the one that generalizes what we do with the distributive law (multiplying term-by-term and collecting terms), the second just multiplies the terms with the matching subscripts.

Definition 4.5.33 (The Cauchy Product) *For* $\sum_{j=0}^{\infty}a_j$ *and* $\sum_{j=0}^{\infty}b_j$, *set*

$$C_k=\sum_{j=0}^{k}a_jb_{k-j}\text{ for each }k\in\mathbb{J}\cup\{0\}.$$

Then $\sum_{k=0}^{\infty}C_k$ *is called the Cauchy product of the given series.*

Definition 4.5.34 (The Hadamard Product) *For* $\sum_{j=0}^{\infty}a_j$ *and* $\sum_{j=0}^{\infty}b_j$, *the series* $\sum_{j=0}^{\infty}a_jb_j$ *is called the Hadamard product of the given series.*

The convergence of two given series does not automatically lead to the convergence of the Cauchy product. The example given in our text (pp 73-74) takes

$$a_j = b_j = \frac{(-1)^j}{\sqrt{j+1}}.$$

We will see in the next section that $\sum_{j=0}^{\infty} a_j$ converges (conditionally). On the other hand, $C_k = \sum_{j=0}^{k} a_j b_{k-j} = (-1)^k \sum_{j=0}^{k} \frac{1}{\sqrt{(k-j+1)(j+1)}}$ is such that

$$|C_k| \geq \sum_{j=0}^{k} \frac{2}{k+2} = (k+1)\frac{2}{k+2}$$

which does not go to zero as k goes to infinity.

If one of the given series is absolutely convergent and the other is convergent we have better news to report.

Theorem 4.5.35 (Mertens Theorem) *For $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$, if (i) $\sum_{j=0}^{\infty} a_j$ converges absolutely, (ii) $\sum_{j=0}^{\infty} a_j = A$, and $\sum_{j=0}^{\infty} b_j = B$, then the Cauchy product of $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$ is convergent to AB.*

Proof. For $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$, let $\{A_n\}$ and $\{B_n\}$ be the respective sequences of nth partial sums. Then

$$C_n = \sum_{k=0}^{n}\left(\sum_{j=0}^{k} a_j b_{n-j}\right) = a_0b_0 + (a_0b_1 + a_1b_0) + \cdots + (a_0b_n + a_1b_{n-1}\cdots + a_nb_0)$$

which can be rearranged–using commutativity, associativity and the distributive laws–to

$$a_0 (b_0 + b_1 + \cdots + b_n) + a_1 (b_0 + b_1 + \cdots + b_{n-1}) + \cdots + a_n b_0.$$

Thus,

$$C_n = a_0 B_n + a_1 B_{n-1} + \cdots + a_{n-1} B_1 + a_n B_0.$$

Since $\sum_{j=0}^{\infty} b_j = B$, for $\beta_n \underset{def}{=} B - B_n$ we have that $\lim_{n\to\infty} \beta_n = 0$. Substitution in the previous equation yields that

$$C_n = a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \cdots + a_{n-1} (B + \beta_1) + a_n (B + \beta_0)$$

which simplifies to

$$C_n = A_n B + (a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-1}\beta_1 + a_n\beta_0).$$

Let

$$\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-1}\beta_1 + a_n\beta_0$$

Because $\lim_{n\to\infty} A_n = A$, we will be done if we can show that $\lim_{n\to\infty} \gamma_n = 0$. In view of the absolute convergence of $\sum_{j=0}^{\infty} a_j$, we can set $\sum_{j=0}^{\infty} |a_j| = \alpha$.

Suppose that $\varepsilon > 0$ is given. From the convergence of $\{\beta_n\}$, there exists a positive integer M such that $n > M$ implies that $|\beta_n| < \varepsilon$. For $n > M$, it follows that

$$|\gamma_n| = |a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-M-1}\beta_{M+1} + a_{n-M}\beta_M + \cdots + a_{n-1}\beta_1 + a_n\beta_0|$$

From the convergence of $\{\beta_n\}$ and $\sum_{j=0}^{\infty} |a_j|$, we have that

$$|a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-M-1}\beta_{M+1}| < \varepsilon\alpha$$

while M being a fixed number and $a_k \to 0$ as $k \to \infty$ yields that

$$|a_{n-M}\beta_M + \cdots + a_{n-1}\beta_1 + a_n\beta_0| \to 0 \text{ as } n \to \infty.$$

Hence, $|\gamma_n| =$ $|a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-M-1}\beta_{M+1} + a_{n-M}\beta_M + \cdots + a_{n-1}\beta_1 + a_n\beta_0|$ implies that $\limsup_{n\to\infty} |\gamma_n| \le \varepsilon\alpha$. Since $\varepsilon > 0$ was arbitrary, it follows that $\lim_{n\to\infty} |\gamma_n| = 0$ as needed. ■

The last theorem in this section asserts that if the Cauchy product of two given convergent series is known to converge and its limit must be the product of the limits of the given series.

Theorem 4.5.36 *If the series* $\sum_{j=0}^{\infty} a_j$, $\sum_{j=0}^{\infty} b_j$, *and* $\sum_{j=0}^{\infty} c_j$ *are known to converge,* $\sum_{j=0}^{\infty} a_j = A$, $\sum_{j=0}^{\infty} b_j = B$, *and* $\sum_{j=0}^{\infty} c_j$ *is the Cauchy product of* $\sum_{j=0}^{\infty} a_j$ *and* $\sum_{j=0}^{\infty} b_j$, *then* $\sum_{j=0}^{\infty} c_j = AB$.

4.5.3 Hadamard Products and Series with Positive and Negative Terms

Notice that $\sum_{j=1}^{\infty} \dfrac{1}{j\,(j+3)^3}$ can be realized as several different Hadamard products; letting $a_j = \dfrac{1}{j}$, $b_j = \dfrac{1}{(j+3)^3}$, $c_j = \dfrac{1}{j\,(j+3)}$ and $d_j = \dfrac{1}{(j+3)^2}$, gives us $\sum_{j=1}^{\infty} \dfrac{1}{j\,(j+3)^3}$ as the Hadamard product of $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ as well as the Hadamard product of $\sum_{j=1}^{\infty} c_j$ and $\sum_{j=1}^{\infty} d_j$. Note that only $\sum_{j=1}^{\infty} a_j$ diverges.

The following theorem offers a useful tool for studying the nth partial sums for Hadamard products.

Theorem 4.5.37 (Summation-by-Parts) *Corresponding to the sequences* $\left\{a_j\right\}_{j=0}^{\infty}$, *let*

$$A_n = \sum_{j=0}^{n} a_j \text{ for } n \in \mathbb{J} \cup \{0\}\,, \text{ and } A_{-1} = 0.$$

Then for the sequence $\left\{b_j\right\}_{j=0}^{\infty}$ *and nonnegative integers* p *and* q *such that* $0 \le p \le q$,

$$\sum_{j=p}^{q} a_j b_j = \sum_{j=p}^{q-1} A_j \left(b_j - b_{j+1}\right) + A_q b_q - A_{p-1} b_p$$

Excursion 4.5.38 *Fill in a proof for the claim.*

As an immediate application of this formula, we can show that the Hadamard product of a series whose *nth* partial sums form a bounded sequence with a series that is generated from a monotonically decreasing sequence of nonnegative terms is convergent.

Theorem 4.5.39 *Suppose that the series* $\sum_{j=0}^{\infty} a_j$ *and* $\sum_{j=0}^{\infty} b_j$ *are such that*

(i) $\left\{\sum_{j=0}^{n} a_j\right\}_{n=0}^{\infty}$ *is a bounded sequence,*

(ii) $\left\{b_j\right\}_{j=0}^{\infty}$ *is a monotonically decreasing sequence of nonnegative reals, and*

(iii) $\lim\limits_{j\to\infty} b_j = 0.$

Then $\sum_{j=0}^{\infty} a_j b_j$ *is convergent.*

Proof. For each $n \in \mathbb{J}$, let $A_n = \sum_{j=0}^{n} a_j$. Then there exists a positive integer M such that $|A_n| \le M$ for all n. Suppose that $\varepsilon > 0$ is given. Because $\left\{b_j\right\}_{j=0}^{\infty}$ is monotonically decreasing to zero, there exists a positive integer K for which $b_K < \dfrac{\varepsilon}{2K}$. Using summation-by-parts, for any integers p and q satisfying $K \le q \le p$, it follows that

$$\begin{aligned}
\left|\sum_{j=p}^{q} a_j b_j\right| &= \left|\sum_{j=p}^{q-1} A_j\left(b_j - b_{j+1}\right) + A_q b_q - A_{p-1} b_p\right| \\
&\le \left|\sum_{j=p}^{q-1} A_j\left(b_j - b_{j+1}\right)\right| + \left|A_q b_q\right| + \left|A_{p-1} b_p\right| \\
&\le \sum_{j=p}^{q-1} \left|A_j\right|\left(b_j - b_{j+1}\right) + \left|A_q\right| b_q + \left|A_{p-1}\right| b_p \\
&\le M\left(\sum_{j=p}^{q-1}\left(b_j - b_{j+1}\right) + b_q + b_p\right) \\
&= M\left(\left(b_p - b_q\right) + b_q + b_p\right) \quad = \quad 2Mb_p \\
&\le 2Mb_K < \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\left\{\sum_{j=0}^{n} a_j b_j\right\}_{n=0}^{\infty}$ is a Cauchy sequence of complex numbers. Therefore, it is convergent. ■

A nice application of this result, gives us an "easy to check" criteria for convergence of series that are generated by sequences with alternating positive and negative terms.

Theorem 4.5.40 (Alternating Series Test) *Suppose that the sequence* $\{u_j\}_{j=1}^{\infty} \subset \mathbb{R}$ *satisfies the following conditions:*

(i) $\operatorname{sgn}(u_j) = -\operatorname{sgn}(u_{j+1})$ *for each* $j \in \mathbb{J} \cup \{0\}$, *where* sgn *denotes "the sign of";*

(ii) $|u_{j+1}| \leq |u_j|$ *for every* j*; and*

(iii) $\lim\limits_{j\to\infty} u_j = 0$.

Then $\sum_{j=1}^{\infty} u_j$ *is convergent. Furthermore, if the sum is denoted by* S*, then* $S_n \leq S \leq S_{n+1}$ *for each* n *where* $\{S_n\}_{n=0}^{\infty}$ *is the sequence of* nth *partial sums.*

The result is an immediate consequence of Theorem 4.5.39; it follows upon setting $a_j = (-1)^j$ and $b_j = \left|c_j\right|$. As an illustration of how "a regrouping argument" can get us to the conclusion, we offer the following proof for your reading pleasure.

Proof. Without loss in generality, we can take $u_0 > 0$. Then $u_{2k+1} < 0$ and $u_{2k} > 0$ for $k = 0, 1, 2, 3, \ldots$ Note that for each $n \in \mathbb{J} \cup \{0\}$,

$$S_{2n} = (u_0 + u_1) + (u_2 + u_3) + \cdots + (u_{2n-2} + u_{2n-1}) + u_{2n}$$

which can be regrouped as

$$S_{2n} = u_0 + (u_1 + u_2) + (u_3 + u_4) + \cdots + (u_{2n-1} + u_{2n}) .$$

The first arrangement justifies that $\{S_{2n}\}_{n=0}^{\infty}$ is monotonically increasing while the second yields that $S_{2n} < u_0$ for each n. By Theorem 4.3.14, the sequence $\{S_{2n}\}_{n=0}^{\infty}$ is convergent. For $\lim\limits_{n\to\infty} S_{2n} = S$, we have that $S_{2n} \leq S$ for each n.

Since $S_{2n-1} = S_{2n} - u_{2n}$, $S_{2n-1} > S_{2n}$ for each $n \in \mathbb{J}$. On the other hand,

$$S_{2n+1} = S_{2n-1} + (u_{2n} + u_{2n+1}) < S_{2n-1}.$$

These inequalities combined with $S_{2n} > S_2 = u_1 + u_2$, yield that the sequence $\{S_{2n-1}\}_{n=1}^{\infty}$ is a monotonically decreasing sequence that is bounded below. Again, by Theorem 4.3.14, $\{S_{2n-1}\}_{n=1}^{\infty}$ is convergent. From (iii), we deduce that $S_{2n-1} \to S$ also. We have that $S_{2n-1} \geq S$ because $\{S_{2n-1}\}_{n=1}^{\infty}$ is decreasing. Pulling this together, leads to the conclusion that $\{S_n\}$ converges to S where $S \leq S_k$ for k odd and $S \geq S_k$ when k is even. ■

Remark 4.5.41 *Combining the Alternating Series Test with Remark 4.5.5 leads to the quick observation that the alternating harmonic series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *is conditionally convergent.*

4.5.4 Discussing Convergence

When asked to discuss the convergence of a given series, there is a system that we should keep in mind. Given the series $\sum_{n=0}^{\infty} u_n$:

1. Check whether or not $\lim_{n\to\infty} u_n = 0$. If not, claim divergence by the kth term test; if yes, proceed to the next step.

2. Check for absolute convergence by testing $\sum_{n=0}^{\infty} |u_n|$. Since $\sum_{n=0}^{\infty} |u_n|$ is a series having nonnegative terms, we have several tests of convergence at our disposal–Comparison, Limit Comparison, Ratio, and Root–in addition to the possibility of recognizing the given series as directly related to a geometric or a p-series. Practice with the tests leads to a better ability to discern which test to use. If $\sum_{n=0}^{\infty} |u_n|$ converges, by any of the our tests, then we conclude that $\sum_{n=0}^{\infty} u_n$ converges absolutely and we are done. If $\sum_{n=0}^{\infty} |u_n|$ diverges by either the Ratio Test or the Root Test, then we conclude that $\sum_{n=0}^{\infty} u_n$ diverges and we are done.

3. If $\sum_{n=0}^{\infty} |u_n|$ diverges by either the Comparison Test or the Limit Comparison Test , then test $\sum_{n=1}^{\infty} u_n$ for conditional convergence–using the Alternating Series Test if it applies. If the series involves nonreal complex terms, try checking the corresponding series of real and imaginary parts.

Excursion 4.5.42 *Discuss the Convergence of each of the following:*

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n-1}}{n^2+1}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n \ln n}{e^n}$

3. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{n+2}$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+\alpha^n}\ , \alpha > -1$

5. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\alpha)}{n^2}$

*******The ratio test leads to the divergence of the first one. The second one is absolutely convergent by the root test. The third one diverges due to failure to pass the kth term test. The behavior of the fourth one depends on α: it diverges for $|\alpha| < 1$ and converges for $|\alpha| > 1$ from the ratio test. Finally the last one converges by comparison.*******

4.5.5 Rearrangements of Series

Given any series $\displaystyle\sum_{j=o}^{\infty}a_j$ and a function $f : \mathbb{J}\cup\{0\} \overset{1-1}{\twoheadrightarrow} \mathbb{J}\cup\{0\}$, the series $\displaystyle\sum_{j=o}^{\infty}a_{f(j)}$ is a **rearrangement** of the original series. Given a series $\displaystyle\sum_{j=o}^{\infty}a_j$ and a rearrangement $\displaystyle\sum_{j=o}^{\infty}a_{f(j)}$, the corresponding sequence of nth partial sums may be completely different. There is no reason to expect that they would have the same limit. The commutative law that works so well for finite sums tells us nothing about what may happen with infinite series. It turns out that if the original series is absolutely convergent, then all rearrangements are convergent to the same limit. In the last section of Chapter 3 in our text, it is shown that the situation is shockingly different for conditionally convergent real series. We will state the result that is proved on pages 76-77 or our text.

Theorem 4.5.43 *Let* $\sum_{j=0}^{\infty} a_j$ *be a real series that converges conditionally. Then for any elements in the extended real number system such that* $-\infty \leq \alpha \leq \beta \leq +\infty$*, there exists a rearrangement of the given series* $\sum_{j=0}^{\infty} a_{f(j)}$ *such that*

$$\liminf_{n\to\infty} \sum_{j=0}^{n} a_{f(j)} = \alpha \qquad \textit{and} \qquad \limsup_{n\to\infty} \sum_{j=0}^{n} a_{f(j)} = \beta.$$

Theorem 4.5.44 *Let* $\sum_{j=0}^{\infty} a_j$ *be a series of complex numbers that converges absolutely. Then every rearrangement of* $\sum_{j=0}^{\infty} a_j$ *converges and each rearrangement converges to the same limit.*

4.6 Problem Set D

1. Use the definition to prove each of the following claims. Your arguments must be well written and make use of appropriate approaches to proof.
   (a) $\lim_{n\to\infty} \frac{n^2 + in}{n^2 + 1} = 1$
   (b) $\lim_{n\to\infty} \frac{3n^2 + i}{2n^3} = 0$
   (c) $\lim_{n\to\infty} \frac{3n + 2}{2n - 1} = \frac{3}{2}$
   (d) $\lim_{n\to\infty} \frac{3n + 1 + 2ni}{n + 3} = 3 + 2i$
   (e) $\lim_{n\to\infty} \frac{1 + 3n}{1 + in} = -3i$

2. Find the limits, if they exist, of the following sequences in $\mathbb{R}^2$. Show enough work to justify your conclusions.

   (a) $\left\{\left(\dfrac{(-1)^n}{n}, \dfrac{\cos n}{n}\right)\right\}_{n=1}^{\infty}$

   (b) $\left\{\left(\dfrac{3n+1}{4n-1}, \dfrac{2n^2+3}{n^2+2}\right)\right\}_{n=1}^{\infty}$

   (c) $\left\{\left(\dfrac{(-1)^n n^2+5}{2n^2}, \dfrac{1+3n}{1+2n}\right)\right\}_{n=1}^{\infty}$

   (d) $\left\{\left(\dfrac{(\sin n)^n}{n}, \dfrac{1}{n^2}\right)\right\}_{n=1}^{\infty}$

   (e) $\left\{\left(\dfrac{\cos n\pi}{n}, \dfrac{\sin(n\pi + (\pi/2))}{n}\right)\right\}_{n=1}^{\infty}$

3. Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to x in Euclidean k-space. Show that $A = \{x_n : n \in \mathbb{J}\} \cup \{x\}$ is closed.

4. For $j, n \in \mathbb{J}$, let $f_j(n) = \dfrac{n^2 \sin\left(\dfrac{\pi j}{4}\right) + 3n}{4j^2n^2 + 2jn + 1}$. Find the limit of the following sequence in $\mathbb{R}^5$, showing enough work to carefully justify your conclusions: $\{(f_1(n), f_2(n), f_3(n), f_4(n), f_5(n))\}_{n=1}^{\infty}$.

5. Find the limit superior and the limit inferior for each of the following sequences.

   (a) $\left\{n \cos \dfrac{n\pi}{2}\right\}_{n=1}^{\infty}$

   (b) $\left\{\dfrac{1+\cos\dfrac{n\pi}{2}}{(-1)^n n^2}\right\}_{n=1}^{\infty}$

   (c) $\left\{\dfrac{1}{2^n} + (-1)^n \cos\dfrac{n\pi}{4} + \sin\dfrac{n\pi}{2}\right\}_{n=1}^{\infty}$

   (d) $\left\{2^{(-1)^n}\left(1+\dfrac{1}{n^2}\right) + 3^{(-1)^{n+1}}\right\}_{n=1}^{\infty}$

6. If $\{a_n\}_{n=0}^{\infty}$ is a bounded sequence of complex numbers and $\{b_n\}_{n=0}^{\infty}$ is a sequence of complex numbers that converges to 0, prove that $\lim\limits_{n\to\infty} a_n b_n = 0$.

7. If $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers with the property that $|a_n - a_{n+1}| \leq \dfrac{1}{2^n}$ for each $n \in \mathbb{J} \cup \{0\}$, prove that $\{a_n\}_{n=0}^{\infty}$ converges.

8. If $\{a_n\}_{n=0}^{\infty}$ is a monotonically increasing sequence such that $a_{n+1} - a_n \leq \dfrac{1}{n}$ for each $n \in \mathbb{J} \cup \{0\}$, must $\{a_n\}_{n=0}^{\infty}$ converge? Carefully justify your response.

9. Discuss the convergence of each of the following. If the given series is convergence and it is possible to find the sum, do so.

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

    (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,(n+2)}$

    (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n}$

    (d) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{n^3}$

    (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

10. Prove the **Limit Comparison Test**.

    Suppose that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are such that $a_n \geq 0$, $b_n \geq 0$ for each $n \in \mathbb{J} \cup \{0\}$, and $\lim\limits_{n\to\infty} a_n \left(b_n\right)^{-1} = L > 0$. Then either $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

    [Hint: For sufficiently large n, justify that $\dfrac{1}{2}L < \dfrac{a_n}{b_n} < \dfrac{3}{2}L$.]

11. Suppose that $a_n \geq 0$ for each $n \in \mathbb{J} \cup \{0\}$.

(a) If $\sum_{n=1}^{\infty} a_n$ converges and $b_n = \sum_{k=n}^{\infty} a_k$, prove that $\sum_{n=1}^{\infty} \left(\sqrt{b_n} - \sqrt{b_{n+1}}\right)$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ diverges and $S_n = \sum_{k=1}^{n} a_k$, prove that $\sum_{n=1}^{\infty} \left(\sqrt{S_{n+1}} - \sqrt{S_n}\right)$ diverges.

12. For each of the following **use our tests** for convergence to check for absolute convergence and, when needed, conditional convergence.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n + i3^n}{5 \cdot 4^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n \sin\left(\frac{(2n-1)\pi}{2}\right)}{n^2+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{2n^2+1} - \sqrt{2n^2-1}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{(n+1)!}$

(e) $\displaystyle\sum_{n=2}^{\infty} (\cos(\pi n)) \left(1 + \frac{1}{n}\right)^{-n^2}$

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{(1+i)^{n+3}}{3^{2n+1} \cdot 4^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \left(\left(\frac{(-1)^n+1}{2}\right)\left(\frac{1+2i}{5}\right)^n + \left(\frac{(-1)^{n+1}+1}{2}\right)\left(\frac{2}{3}\right)^n\right)$

13. Justify that $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right)^p$ is absolutely convergent for $p > 2$, conditionally convergent for $0 < p \leq 2$, and divergent for $p \leq 0$.

14. Let ℓ_2 be the collection of infinite sequences $\{x_n\}_{n=1}^{\infty}$ of reals such that $\sum_{n=1}^{\infty} x_n^2$

converges and define $d\left(x,y\right)=\sqrt{\sum_{n=1}^{\infty}\left(x_n-y_n\right)^2}$ for each $x=\{x_n\}_{n=1}^{\infty}, y=\{y_n\}_{n=1}^{\infty}\in\ell_2$. Show that (ℓ_2,d) is a metric space.

15. A sequence $\{x_n\}_{n=1}^{\infty}$ of reals is bounded if and only if there is a number m such that $|x_n|\le m$ for each $n\in\mathbb{J}$. Let M denote the collection of all bounded sequences, and defined $d\left(x,y\right)=\sup_{n\in\mathbb{J}}|x_n-y_n|$. Show that (M,d) is a metric space.

16. Let B be the collection of all absolutely convergent series and define $d\left(x,y\right)=\sum_{n=1}^{\infty}|x_n-y_n|$. Show that (B,d) is a metric space.

Chapter 5

Functions on Metric Spaces and Continuity

When we studied real-valued functions of a real variable in calculus, the techniques and theory built on properties of continuity, differentiability, and integrability. All of these concepts are defined using the precise idea of a limit. In this chapter, we want to look at functions on metric spaces. In particular, we want to see how mapping metric spaces to metric spaces relates to properties of subsets of the metric spaces.

5.1 Limits of Functions

Recall the definitions of limit and continuity of real-valued functions of a real variable.

Definition 5.1.1 *Suppose that f is a real-valued function of a real variable, $p \in \mathbb{R}$, and there is an interval I containing p which, except possibly for p is in the domain of f. Then* ***the limit of*** f ***as*** x ***approaches*** p *is L if and only if*

$$(\forall \varepsilon)\,(\varepsilon > 0 \Rightarrow (\exists \delta = \delta(\varepsilon))\,(\delta > 0 \wedge (\forall x)\,(0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon)))\,.$$

In this case, we write $\lim_{x \to p} f(x) = L$ which is read as "the limit of f of x as x approaches p is equal to L."

Definition 5.1.2 *Suppose that f is a real-valued function of a real variable and $p \in \operatorname{dom}(f)$. Then f is* ***continuous at*** *p if and only if $\lim_{x \to p} f(x) = f(p)$.*

These are more or less the way limit of a function and continuity of a function at a point were defined at the time of your first encounter with them. With our new terminology, we can relax some of what goes into the definition of limit. Instead of going for an interval (with possibly a point missing), we can specify that the point p be a limit point of the domain of f and then insert that we are only looking at the real numbers that are both in the domain of the function and in the open interval. This leads us to the following variation.

Definition 5.1.3 *Suppose that f is a real-valued function of a real variable, $\operatorname{dom}(f) = A$, and $p \in A'$ (i.e., p is a limit point of the domain of f). Then* ***the limit of f as x approaches p*** *is L if and only if*

$$(\forall \varepsilon)\,(\varepsilon > 0 \Rightarrow (\exists \delta = \delta(\varepsilon) > 0)\;\big[(\forall x)\,(x \in A \wedge 0 < |x-p| < \delta \Rightarrow |f(x) - L| < \varepsilon)\big])$$

Example 5.1.4 *Use the definition to prove that* $\lim\limits_{x \to 3} \left(2x^2 + 4x + 1\right) = 31$.

Before we offer a proof, we'll illustrate some "expanded "scratch work that leads to the information needed in order to offer a proof. We want to show that, corresponding to each $\varepsilon > 0$ we can find a $\delta > 0$ such that $0 < |x-3| < \delta \Rightarrow \left|\left(2x^2+4x+1\right) - 31\right| < \varepsilon$. The easiest way to do this is to come up with a δ that is a function of ε. Note that

$$\left|\left(2x^2+4x+1\right) - 31\right| = \left|2x^2+4x-30\right| = 2\,|x-3|\,|x+5|\,.$$

The $|x-3|$ is good news because it is ours to make as small as we choose. But if we restrict $|x-3|$ there is a corresponding restriction on $|x+5|$; to take care of this part we will put a cap on δ which will lead to simpler expressions. Suppose that we place a 1^{st} restriction on δ of requiring that $\delta \le 1$. If $\delta \le 1$, then $0 < |x-3| < \delta \le 1 \Rightarrow |x+5| = |(x-3)+8| \le |x-3| + 8 < 9$. Now

$$\left|\left(2x^2+4x+1\right) - 31\right| = 2\,|x-3|\,|x+5| < 2 \cdot \delta \cdot 9 \le \varepsilon$$

whenever $\delta \le \dfrac{\varepsilon}{18}$. To get both bounds to be in effect we will take $\delta = \max\left\{1, \dfrac{\varepsilon}{18}\right\}$. This concludes that "expanded "scratch work.

Proof. *For $\varepsilon > 0$, let $\delta = \max\left\{1, \dfrac{\varepsilon}{18}\right\}$. Then*

$$0 < |x-3| < \delta \le 1 \Rightarrow |x+5| = |(x-3)+8| \le |x-3| + 8 < 9$$

and

$$\left|\left(2x^2+4x+1\right)-31\right| = 2\,|x-3|\,|x+5| < 2\cdot\delta\cdot 9 \le 18\cdot\frac{\varepsilon}{18} = \varepsilon.$$

Since $\varepsilon > 0$ *was arbitrary, we conclude that, for every* $\varepsilon > 0$, *there exists a* $\delta = \min\left\{1, \dfrac{\varepsilon}{18}\right\} > 0$, *such that* $0 < |x-3| < \delta \Rightarrow \left|\left(2x^2+4x+1\right)-31\right| < \varepsilon$*; i.e.,* $\lim\limits_{x\to 3}\left(2x^2+4x+1\right) = 31$. ■

Excursion 5.1.5 *Use the definition to prove that* $\lim\limits_{x\to 1}\left(x^2+5x\right) = 6$.

Space for scratch work.

A Proof.

For this one, the δ that you define will depend on the nature of the first restriction that you placed on δ in order to obtain a nice upper bound on $|x+6|$; if you chose $\delta \le 1$ as your first restriction, then $\delta = \min\left\{1, \dfrac{\varepsilon}{8}\right\}$ would have been what worked in the proof that was offered.

You want to be careful not to blindly take $\delta \le 1$ as the first restriction. For example, if you are looking at the greatest integer function as $x \to \frac{1}{2}$, you would need to make sure that δ never exceeded $\frac{1}{2}$ in order to stay away from the nearest "jumps"; if you have a rational function for which $\frac{1}{2}$ is a zero of the denominator and you are looking at the limit as $x \to \frac{1}{4}$, then you couldn't let δ be as great at $\frac{1}{4}$

so you might try taking $\delta \leq \frac{1}{6}$ as a first restriction. Our next example takes such a consideration into account.

Example 5.1.6 *Use the definition to prove that* $\lim_{x\to -1} \frac{x^2+3}{2x+1} = -4$.

Space for scratch work.

Proof. *For* $\varepsilon > 0$, *let* $\delta = \min\left\{\frac{1}{4}, \frac{2\varepsilon}{25}\right\}$. *From* $0 < |x+1| < \delta \leq \frac{1}{4}$, *we have that*

$$|x+7| = |(x+1)+6| < |x+1| + 6 < \frac{1}{4} + 6 = \frac{25}{4}$$

and

$$|2x+1| = 2\left|x+\frac{1}{2}\right| = 2\left|(x+1)-\frac{1}{2}\right| \geq 2\left||x+1|-\frac{1}{2}\right| > 2\left(\frac{1}{4}\right) = \frac{1}{2}.$$

Furthermore,

$$\left|\left(\frac{x^2+3}{2x+1}\right)-(-4)\right| = \left|\frac{x^2+8x+7}{2x+1}\right| = \frac{|x+1|\,|x+7|}{|2x+1|} < \frac{\delta \cdot \left(\frac{25}{4}\right)}{\frac{1}{2}} =$$

$$\frac{25\cdot\delta}{2} \leq \frac{25}{2}\cdot\frac{2\varepsilon}{25} = \varepsilon.$$

Since $\varepsilon > 0$ *was arbitrary, for every* $\varepsilon > 0$ *there exists a* $\delta = \min\left\{\frac{1}{4}, \frac{2\varepsilon}{25}\right\} > 0$ *such that* $0 < |x+1| < \delta$ *implies that* $\left|\left(\frac{x^2+3}{2x+1}\right)-(-4)\right| < \varepsilon$*; that is,* $\lim_{x\to -1}\frac{x^2+3}{2x+1} = -4$. ■

In Euclidean $\mathbb{R}$ space, the metric is realized as the absolute value of the difference. Letting d denote this metric allows us to restate the definition of $\lim_{x \to p} f(x) = L$ as

$$(\forall \varepsilon)(\varepsilon > 0 \Rightarrow (\exists \delta = \delta(\varepsilon) > 0) \\ \left[(\forall x)(x \in A \wedge 0 < d(x,p) < \delta \Rightarrow d(f(x), L) < \varepsilon)\right]).$$

Of course, at this point we haven't gained much; this form doesn't look particularly better than the one with which we started. On the other hand, it gets us nearer to where we want to go which is to the limit of a function that is from one metric space to another–neither of which is $\mathbb{R}^1$. As a first step, let's look at the definition when the function is from an arbitrary metric space into $\mathbb{R}^1$. Again we let d denote the Euclidean $1-$metric.

Definition 5.1.7 *Suppose that A is a subset of a metric space (S, d_S) and that f is a function with domain A and range contained in $\mathbb{R}^1$; i.e., $f : A \to \mathbb{R}^1$. then "f tends to L as x tends to p* ***through points of*** *A" if and only if*

(i) *p is a limit point of A, and*

(ii) $(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon) > 0)((\forall x)(x \in A \wedge 0 < d_S(x,p) < \delta$ $\Rightarrow d(f(x), L) < \varepsilon))$.

In this case, we write $f(x) \to L$ as $x \to p$ for $x \in A$, or $f(x) \to L$ as $x \underset{A}{\to} p$, or

$$\lim_{\substack{x \to p \\ x \in A}} f(x) = L.$$

Example 5.1.8 *Let $f : \mathbb{C} \to \mathbb{R}$ be given by $f(z) = \mathrm{Re}(z)$. Prove that*

$$\lim_{\substack{z \to 3+i \\ z \in \mathbb{C}}} f(z) = 3.$$

Space for scratch work.

For this one, we will make use of the fact that for any complex number ζ, $|\mathrm{Re}\,(\zeta)| \leq |\zeta|$.

Proof. *For* $\varepsilon > 0$, *let* $\delta = \varepsilon$. *Then* $0 < |z - (3+i)| < \delta = \varepsilon$ *implies that*

$$|f(z) - 3| = |\mathrm{Re}\,(z) - 3| = |\mathrm{Re}\,(z - (3+i))| \leq |z - (3+i)| < \varepsilon.$$

Since $\varepsilon > 0$ *was arbitrary, we conclude that* $\lim\limits_{\substack{z \to 3+i \\ z \in \mathbb{C}}} f(z) = 3$. ■

Remark 5.1.9 *Notice that, in the definition of* $\lim\limits_{\substack{x \to p \\ x \in A}} f(x) = L$, *there is neither a requirement that* f *be defined at* p *nor an expectation that* p *be an element of* A. *Also, while it isn't indicated, the* $\delta > 0$ *that is sought may be dependent on* p.

Finally we want to make the transition to functions from one arbitrary metric space to another.

Definition 5.1.10 *Suppose that* A *is a subset of a metric space* (S, d_S) *and that* f *is a function with domain* A *and range contained in a metric space* (X, d_X)*; i.e.,* $f : A \to X$. *Then "*f *tends to* L *as* x *tends to* p ***through points of*** A*" if and only if*

(i) p *is a limit point of* A, *and*

(ii) $(\forall \varepsilon > 0)\,(\exists \delta = \delta(\varepsilon) > 0)\,((\forall x)\,(x \in A \wedge 0 < d_S(x, p) < \delta$

$$\Rightarrow d_X(f(x), L) < \varepsilon)).$$

In this case, we write $f(x) \to L$ *as* $x \to p$ *for* $x \in A$, *or* $f(x) \to L$ *as* $x \underset{A}{\to} p$, *or*

$$\lim_{\substack{x \to p \\ x \in A}} f(x) = L.$$

Example 5.1.11 *For* $p \in \mathbb{R}^1$*, let* $f(p) = (2p+1, p^2)$*. Then* $f : \mathbb{R}^1 \longrightarrow \mathbb{R}^2$*. Use the definition of limit to from that* $\lim_{p \to 1} f(p) = (3, 1)$ *with respect to the Euclidean metrics on each space.*

Space for scratch work.

Proof. *For* $\varepsilon > 0$*, let* $\delta = \min\left\{1, \dfrac{\varepsilon}{\sqrt{13}}\right\}$*. Then* $0 < d_{\mathbb{R}}(p,1) = |p-1| < \delta \leq 1$ *implies that*

$$|p+1| = |(p-1)+2| \leq |p-1| + 2 < 3$$

and

$$\sqrt{4+(p+1)^2} < \sqrt{4+9} = \sqrt{13}.$$

Hence, for $0 < d_{\mathbb{R}}(p,1) = |p-1| < \delta$*,*

$$\begin{aligned} d_{\mathbb{R}^2}\left(f(p), (3,1)\right) &= \sqrt{((2p+1)-3)^2 + \left(p^2-1\right)^2} \\ &= |p-1|\sqrt{4+(p+1)^2} < \delta \cdot \sqrt{13} \leq \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ *was arbitrary, we conclude that* $\lim_{p \to 1}(2p+1, p^2) = (3,1)$*.* ■

Remark 5.1.12 *With few exceptions our limit theorems for functions of real-valued functions of a real variable that involved basic combinations of functions have direct, straightforward analogs to functions on an arbitrary metric spaces. Things can get more difficult when we try for generalizations of results that involved comparing function values. For the next couple of excursions, you are just being asked to practice translating results from one setting to our new one.*

Excursion 5.1.13 *Let* A *be a subset of a metric space* S *and suppose that* $f : A \to \mathbb{R}^1$ *is given. If*

$$f \to L \text{ as } p \to p_0 \text{ in } A \text{ and } f \to M \text{ as } p \to p_0 \text{ in } A$$

prove that $L = M$. After reading the following proof for the case of real-valued functions of a real variable, use the space provided to write a proof for the new setting.

Proof. We want to prove that, if $f \to L$ as $x \to a$ and $f \to M$ as $x \to a$, then $L = M$. For $L \neq M$, let $\epsilon = \frac{1}{2} \cdot |L - M|$. By the definition of limit, there exists positive numbers δ_1 and δ_2 such that $0 < |x - a| < \delta_1$ implies $|f(x) - L| < \epsilon$ and $0 < |x - a| < \delta_2$ implies $|f(x) - M| < \epsilon$. Choose $x_0 \in \mathbb{R}$ such that $0 < |x_0 - a| < \min\{\delta_1, \delta_2\}$. Then $|L - M| \leq |L - f(x_0)| + |M - f(x_0)| < 2\epsilon$ which contradicts the trichotomy law. ■

Excursion 5.1.14 *Let f and g be real-valued functions with domain A, a subset of a metric space (S, d). If $\lim\limits_{\substack{p \to p_0 \\ p \in A}} f(p) = L$ and $\lim\limits_{\substack{p \to p_0 \\ p \in A}} g(p) = M$, then $\lim\limits_{\substack{p \to p_0 \\ p \in A}} (f + g)(p) = L + M$.*

After reading the following proof for the case of real-valued functions of a real variable, use the space provided to write a proof for the new setting.

Proof. We want to show that, if $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then $\lim\limits_{x \to a} (f + g)(x) = L + M$. Let $\epsilon > 0$ be given. Then there exists positive numbers δ_1 and δ_2 such that $0 < |x - a| < \delta_1$ implies $|f(x) - L| < \epsilon/2$ and $0 < |x - a| < \delta_2$ implies $|g(x) - M| < \epsilon/2$. For $\delta = \min\{\delta_1, \delta_2\}$, $0 < |x - a| < \delta$ implies that

$|(f+g)(x)-(L+M)| \leq |f(x)-L| + |g(x)-M| < \epsilon.$ ■

Theorem 4.1.17 gave us a characterization of limit points in terms of limits of sequences. This leads nicely to a characterization of limits of functions in terms of behavior on convergent sequences.

Theorem 5.1.15 (Sequences Characterization for Limits of Functions) *Suppose that* (X, d_X) *and* (Y, d_Y) *are metric spaces,* $E \subset X$, $f : E \longrightarrow Y$ *and* p *is a limit point of* E. *Then* $\lim_{\substack{x \to p \\ x \in E}} f(x) = q$ *if and only if*

$$(\forall \{p_n\}) \left[\left(\{p_n\} \subset E \wedge (\forall n)\,(p_n \neq p) \wedge \lim_{n\to\infty} p_n = p \right) \Rightarrow \lim_{n\to\infty} f(p_n) = q \right].$$

Excursion 5.1.16 *Fill in what is missing in order to complete the following proof of the theorem.*

Proof. Let X, Y, E, f, and p be as described in the introduction to Theorem 5.1.15. Suppose that $\lim_{\substack{x \to p \\ x \in E}} f(x) = q$. Since p is a limit point of E, by Theorem $\underset{(1)}{\underline{\hspace{1.5cm}}}$, there exists a sequence $\{p_n\}$ of elements in E such that $p_n \neq p$ for all $n \in \mathbb{J}$, and $\underset{(2)}{\underline{\hspace{6cm}}}$. For $\varepsilon > 0$, because $\lim_{\substack{x \to p \\ x \in E}} f(x) = q$, there exists $\delta > 0$ such that $0 < d_X(x, p) < \delta$ and $x \in E$ implies that $\underset{(3)}{\underline{\hspace{3cm}}}$. From $\lim_{n\to\infty} p_n = p$ and $p_n \neq p$, we also know that there exists a positive integer M such that $n > M$ implies that $\underset{(4)}{\underline{\hspace{6cm}}}$. Thus, it follows that

$d_Y\left(f\left(p_n\right), q\right) < \varepsilon$ for all $n > M$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim\limits_{n\to\infty} f\left(p_n\right) = q$. Finally, because $\{p_n\} \subset E$ was arbitrary, we have that

$$(\forall\{p_n\})\left[\left(\underset{(5)}{\underline{\hspace{8cm}}}\right) \Rightarrow \lim_{n\to\infty} f\left(p_n\right) = q\right].$$

We will give a proof by contrapositive of the converse. Suppose that $\lim\limits_{\substack{x\to p\\ x\in E}} f(x) \neq q$. Then there exists a positive real number ε such that corresponding to each positive real number δ there is a point $x_\delta \in E$ for which $0 < d_X\left(x_\delta, p\right) < \delta$ and $d_Y\left(f\left(x_\delta\right), q\right) \geq \varepsilon$. In particular, for each $n \in \mathbb{J}$, corresponding to $\dfrac{1}{n}$ there is a point $p_n \in E$ such that $\underset{(6)}{\underline{\hspace{3cm}}}$ and $d_Y\left(f\left(p_n\right), q\right) \geq \varepsilon$. Hence, $\lim\limits_{n\to\infty} f\left(p_n\right) \neq q$. Thus, there exists a sequence $\{p_n\} \subset E$ such that $\lim\limits_{n\to\infty} p_n = p$ and $\underset{(7)}{\underline{\hspace{3cm}}}$; i.e.,

$$(\exists\{p_n\})\left[\left(\{p_n\} \subset E \wedge (\forall n)\left(p_n \neq p\right) \wedge \lim_{n\to\infty} p_n = p\right) \wedge \lim_{n\to\infty} f\left(p_n\right) \neq q\right]$$

which is equivalent to

$$\neg(\forall\{p_n\})\left[\left(\{p_n\} \subset E \wedge (\forall n)\left(p_n \neq p\right) \wedge \lim_{n\to\infty} p_n = p\right) \Rightarrow \lim_{n\to\infty} f\left(p_n\right) = q\right].$$

Therefore, we have shown that $\lim\limits_{\substack{x\to p\\ x\in E}} f(x) \neq q$ implies that

$$\neg(\forall\{p_n\})\left[\left(\{p_n\} \subset E \wedge (\forall n)\left(p_n \neq p\right) \wedge \lim_{n\to\infty} p_n = p\right) \Rightarrow \lim_{n\to\infty} f\left(p_n\right) = q\right].$$

Since the $\underset{(8)}{\underline{\hspace{8cm}}}$ is logically equivalent to the converse, this concludes the proof. ■

Acceptable responses are: (1) 4.1.17,(2) $\lim\limits_{n\to\infty} p_n = p$, (3) $d_Y\left(f(x), q\right) < \varepsilon$, (4) $0 < d_X\left(p_n, p\right) < \delta$, (5) $\{p_n\} \subset E \wedge (\forall n)\left(p_n \neq p\right) \wedge \lim\limits_{n\to\infty} p_n = p$, (6) $0 < d_X\left(p_n, p\right) < \dfrac{1}{n}$, (7) $\lim\limits_{n\to\infty} f\left(p_n\right) \neq q$, (8) contrapositive.

The following result is an immediate consequence of the theorem and Lemma 4.1.7.

Corollary 5.1.17 *Limits of functions on metric spaces are unique.*

Remark 5.1.18 *In view of Theorem 5.1.15, functions from metric spaces into subsets of the complex numbers will satisfy the "limits of combinations" properties of sequences of complex numbers that were given in Theorem 4.3.2. For completeness, we state it as a separate theorem.*

Theorem 5.1.19 *Suppose that* (X, d_X) *is a metric space,* $E \subset X$*,* p *is a limit point of* E*,* $f : E \longrightarrow \mathbb{C}$*,* $g : E \longrightarrow \mathbb{C}$*,* $\lim_{\substack{x \to p \\ x \in E}} f(x) = A$*, and* $\lim_{\substack{x \to p \\ x \in E}} g(x) = B$*. Then*

(a) $\lim_{\substack{x \to p \\ x \in E}} (f + g)(x) = A + B$

(b) $\lim_{\substack{x \to p \\ x \in E}} (fg)(x) = AB$

(c) $\lim_{\substack{x \to p \\ x \in E}} \frac{f}{g}(x) = \frac{A}{B}$ *whenever* $B \neq 0$.

While these statements are an immediate consequence of Theorem 4.3.2 and Theorem 5.1.15 completing the following excursions can help you to learn the approaches to proof. Each proof offered is independent of Theorems 4.3.2 and Theorem 5.1.15.

Excursion 5.1.20 *Fill in what is missing to complete a proof of Theorem 5.1.19(a).*

Proof. Suppose $\varepsilon > 0$ is given. Because $\lim_{\substack{x \to p \\ x \in E}} f(x) = A$, there exists a positive real δ_1 such that $x \in E$ and $0 < d_X(x, p) < \delta_1$ implies that $|f(x) - A| < \frac{\varepsilon}{2}$. Since $\underset{(1)}{\underline{\qquad\qquad}}$, there exists a positive real number δ_2 such that $x \in E$ and $0 < d_X(x, p) < \delta_2$ implies that $|g(x) - B| < \frac{\varepsilon}{2}$. Let $\delta = \underset{(2)}{\underline{\qquad\qquad}}$. It follows from the triangular inequality that, if $x \in E$ and $0 < d_X(x, p) < \delta$, then

$$\begin{aligned} |(f+g)(x) - (A+B)| &= \left| (f(x) - A) + \left(\underset{(3)}{\underline{\qquad\qquad}} \right) \right| \\ &\leq \underset{(4)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}} < \underset{(5)}{\underline{\quad}}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{\substack{x\to p\\ x\in E}} (f+g)(x) = A + B$ as claimed.

■

Acceptable responses are: (1) $\lim_{\substack{x\to p\\ x\in E}} g(x) = B$, (2) $\min\{\delta_1, \delta_2\}$, (3) $(g(x) - B)$, (4) $|f(x) - A| + |g(x) - B|$, (5) ε

Excursion 5.1.21 *Fill in what is missing to complete a proof of Theorem 5.1.19(b).*

Proof. Because $\lim_{\substack{x\to p\\ x\in E}} f(x) = A$, there exist a positive real number δ_1 such that $x \in E$ and $0 < d_X(x, p) < \delta_1$ implies that $|f(x) - A| < 1$; i.e., $|f(x)| - |A| < 1$. Hence, $|f(x)| < 1 + |A|$ for all $x \in E$ such that $0 < d_X(x, p) < \delta_1$.

Suppose that $\varepsilon > 0$ is given. If $B = 0$, then $\lim_{\substack{x\to p\\ x\in E}} g(x) = 0$ yields the existence of a positive real number δ_2 such that $x \in E$ and $0 < d_X(x, p) < \delta_2$ implies that

$$|g(x)| < \frac{\varepsilon}{1 + |A|}.$$

Then for $\delta^* = \underset{(1)}{\underline{\hspace{3cm}}}$, we have that

$$|(fg)(x)| = |f(x)|\,|g(x)| < (1 + |A|) \cdot \underset{(2)}{\underline{\hspace{3cm}}}.$$

Hence, $\lim_{\substack{x\to p\\ x\in E}} (fg)(x) = AB = 0$. Next we suppose that $B \neq 0$. Then there exists a positive real numbers δ_3 and δ_4 for which $|f(x) - A| < \frac{\varepsilon}{2|B|}$ and $|g(x) - B| < \frac{\varepsilon}{2(1 + |A|)}$ whenever $0 < d_X(x, p) < \delta_3$ and $0 < d_X(x, p) < \delta_4$, respectively, for $x \in E$. Now let $\delta = \min\{\delta_1, \delta_3, \delta_4\}$. It follows that if $x \in E$ and $0 < d_X(x, p) <$

δ, then

$$\begin{aligned} |(fg)(x) - AB| &= \left| f(x)\,g(x) - \underset{(3)}{\underline{\qquad\quad}} + \underset{(4)}{\underline{\qquad\quad}} - AB \right| \\ &\le |f(x)|\,|g(x) - B| + \underset{(5)}{\underline{\qquad\qquad\quad}} \\ &< (1 + |A|)\,|g(x) - B| + |B| \left| \underset{(6)}{\underline{\qquad\qquad\quad}} \right| \\ &< \underset{(7)}{\underline{\qquad\qquad\qquad\qquad\qquad}} = \underset{(8)}{\underline{\qquad\qquad\quad}}. \end{aligned}$$

Again, since $\varepsilon > 0$ was arbitrary, we conclude that $\lim\limits_{\substack{x \to p \\ x \in E}} (fg)(x) = AB$ as needed. ■

Acceptable responses include: (1) $\min\{\delta_1, \delta_2\}$, (2) $\varepsilon\,(1 + |A|)^{-1}$, (3) $f(x)\,B$, (4) $f(x)\,B$, (5) $|B|\,|f(x) - A|$, (6) $|f(x) - A|$, (7) $(1 + |A|)\dfrac{\varepsilon}{2\,(1 + |A|)} + |B|\dfrac{\varepsilon}{2\,|B|}$ (8) ε.

Excursion 5.1.22 *Fill in what is missing to complete a proof of Theorem 5.1.19(c).*

Proof. In view of Theorem 5.1.19(b), it will suffice to prove that, under the given hypotheses, $\lim\limits_{\substack{x \to p \\ x \in E}} \dfrac{1}{g(x)} = \dfrac{1}{B}$. First, we will show that, for $B \neq 0$, the modulus of g is bounded away from zero. Since $|B| > 0$ and $\lim\limits_{\substack{x \to p \\ x \in E}} g(x) = B$, there exists a positive real number $\delta_1 > 0$ such that $x \in E$ and $\underset{(1)}{\underline{\qquad\qquad\quad}}$ implies that $|g(x) - B| < \dfrac{|B|}{2}$. It follows from the (other) $\underset{(2)}{\underline{\qquad\qquad\qquad\qquad\qquad}}$ that, if $x \in E$ and $0 < d_X(x, p) < \delta_1$, then

$$\begin{aligned} |g(x)| = |(g(x) - B) + B| \ge \left| |g(x) - B| \;\underline{\qquad\qquad\quad} \right| \\ > \frac{|B|}{2}. \end{aligned}$$

Suppose that $\varepsilon > 0$ is given. Then $\dfrac{|B|^2 \varepsilon}{2} > 0$ and $\lim\limits_{\substack{x \to p \\ x \in E}} g(x) = B$ yields the existence of a positive real number δ_2 such that $|g(x) - B| < \dfrac{|B|^2 \varepsilon}{2}$ whenever $x \in E$ and $0 < d_X(x, p) < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $x \in E$ and $0 < d_X(x, p) < \delta$ we have that

$$\left| \frac{1}{g(x)} - \frac{1}{B} \right| = \frac{|g(x) - B|}{|B|\,|g(x)|} < \underset{(3)}{\underline{\hspace{3cm}}} = \underset{(4)}{\underline{\hspace{1.5cm}}}.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\underset{(5)}{\underline{\hspace{3cm}}}$.

Finally, letting $h(x) = \dfrac{1}{g(x)}$, by Theorem $\underset{(6)}{\underline{\hspace{1.5cm}}}$,

$$\lim_{\substack{x \to p \\ x \in E}} \frac{f}{g}(x) = \lim_{\substack{x \to p \\ x \in E}} f(x) \underset{(7)}{\underline{\hspace{1.5cm}}} = \underset{(8)}{\underline{\hspace{1.5cm}}}.$$

■

Acceptable responses are: (1) $0 < d_X(x, p) < \delta_1$, (2) triangular inequality (3) $\dfrac{\varepsilon |B|^2}{2|B|\left(\dfrac{|B|}{2}\right)}$, (4) ε (5) $\lim\limits_{\substack{x \to p \\ x \in E}} \dfrac{1}{g(x)} = \dfrac{1}{B}$, (6) 5.1.19(b), (7) $h(x)$, (8) $A \cdot \dfrac{1}{B}$.

From Lemma 4.3.1, it follows that the limit of the sum and the limit of the product parts of Theorem 5.1.19 carry over to the sum and inner product of functions from metric spaces to Euclidean $k - space$.

Theorem 5.1.23 *Suppose that X is a metric space, $E \subset X$, p is a limit point of E, $\mathbf{f} : E \longrightarrow \mathbb{R}^k$, $\mathbf{g} : E \longrightarrow \mathbb{R}^k$, $\lim\limits_{\substack{x \to p \\ x \in E}} f(x) = \mathbf{A}$, and $\lim\limits_{\substack{x \to p \\ x \in E}} g(x) = \mathbf{B}$. Then*

(a) $\lim\limits_{\substack{x \to p \\ x \in E}} (\mathbf{f} + \mathbf{g})(x) = \mathbf{A} + \mathbf{B}$ *and*

(b) $\lim\limits_{\substack{x \to p \\ x \in E}} (\mathbf{f} \bullet \mathbf{g})(x) = \mathbf{A} \bullet \mathbf{B}$

In the set-up of Theorem 5.1.23, note that $\mathbf{f}+\mathbf{g} : E \longrightarrow \mathbb{R}^k$ while $\mathbf{f} \bullet \mathbf{g} : E \longrightarrow \mathbb{R}$.

5.2 Continuous Functions on Metric Spaces

Recall that in the case of real-valued functions of a real variable getting from the general idea of a functions having limits to being continuous simply added the property that the values approached are actually the values that are achieved. There is nothing about that transition that was tied to the properties of the reals. Consequently, the definition of continuous functions on arbitrary metric spaces should come as no surprise. On the other hand, an extra adjustment is needed to allow for the fact that we can consider functions defined at isolated points of subsets of metric spaces.

Definition 5.2.1 *Suppose that* (X, d_X) *and* (Y, d_Y) *are metric spaces,* $E \subset X$*,* $f : E \longrightarrow Y$ *and* $p \in E$*. Then* f *is* ***continuous at*** p *if and only if*

$$(\forall \varepsilon > 0)\,(\exists \delta = \delta(\varepsilon) > 0)\left[(\forall x)\,(x \in E \wedge d_X(x, p) < \delta) \Rightarrow d_Y(f(x), f(p)) < \varepsilon\right].$$

***Theorem* 5.2.2** *Suppose that* (X, d_X) *and* (Y, d_Y) *are metric spaces,* $E \subset X$*,* $f : E \longrightarrow Y$ *and* $p \in E$ *and* p *is a limit point of* E*. Then* f *is continuous at* p *if and only if* $\lim\limits_{\substack{x \to p \\ x \in E}} f(x) = f(p)$

Definition 5.2.3 *Suppose that* (X, d_X) *and* (Y, d_Y) *are metric spaces,* $E \subset X$ *and* $f : E \longrightarrow Y$*. Then* f *is* ***continuous on*** E *if and only if* f *is continuous at each* $p \in E$*.*

Remark 5.2.4 *The property that was added in order to get the characterization that is given in Theorem 5.2.2 was the need for the point to be a limit point. The definition of continuity at a point is satisfied for isolated points of* E *because each isolated point* p *has the property that there is a neighborhood of* p*,* $N_{\delta^*}(p)$*, for which* $E \cap N_{\delta^*}(p) = \{p\}$*; since* $p \in \operatorname{dom}(f)$ *and* $d_X(p, p) = d_Y(f(p), f(p)) = 0$*, we automatically have that* $(\forall x)\,(x \in E \wedge d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$ *for any* $\varepsilon > 0$ *and any positive real number* δ *such that* $\delta < \delta^*$*.*

Remark 5.2.5 *It follows immediately from our limit theorems concerning the algebraic manipulations of functions for which the limits exist, the all real-valued polynomials in k real variables are continuous in* $\mathbb{R}^k$*.*

Remark 5.2.6 *Because* $f(1) = (3, 1)$ *for the* $f(p) = \left(\left(2p + 1, p^2\right)\right) : \mathbb{R}^1 \longrightarrow \mathbb{R}^2$ *that was given in Example 5.1.11, our work for the example allows us to claim that* f *is continuous at* $p = 1$*.*

Theorem 5.1.15 is not practical for use to show that a specific function is continuous; it is a useful tool for proving some general results about continuous functions on metric spaces and can be a nice way to show that a given function is not continuous.

Example 5.2.7 *Prove that the function* $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *given by* $f((x,y)) = \begin{cases} \dfrac{xy}{x^3+y^3} & , \text{ for } (x,y) \neq (0,0) \\ 0 & , \text{ for } x = y = 0 \end{cases}$ *is not continuous at* $(0,0)$.

Let $p_n = \left(\frac{1}{n}, \frac{1}{n}\right)$. *Then* $\{p_n\}_{n=1}^{\infty}$ *converges to* $(0,0)$, *but*

$$\lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)^3 + \left(\frac{1}{n}\right)^3} = \lim_{n\to\infty} \frac{n}{2} = +\infty \neq 0.$$

Hence, by the Sequences Characterization for Limits of Functions, we conclude that the given f *is not continuous at* $(0,0)$.

Example 5.2.8 *Use the definition to prove that* $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *given by* $f((x,y)) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & , \text{ for } (x,y) \neq (0,0) \\ 0 & , \text{ for } x = y = 0 \end{cases}$ *is continuous at* $(0,0)$.

We need to show that $\lim_{(x,y)\to(0,0)} f((x,y)) = 0$. *Because the function is defined in two parts, it is necessary to appeal to the definition. For* $\varepsilon > 0$, *let* $\delta = \varepsilon$. *Then*

$$0 < d_{\mathbb{R}\times\mathbb{R}}((x,y),(0,0)) = \sqrt{x^2+y^2} < \delta = \varepsilon$$

implies that

$$|f((x,y)) - 0| = \left|\frac{x^2y}{x^2+y^2}\right| \leq \frac{\left(x^2+y^2\right)|y|}{x^2+y^2} = |y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} < \varepsilon.$$

Because $\varepsilon > 0$ *was arbitrary, we conclude that* $\lim_{(x,y)\to(0,0)} f((x,y)) = 0 = f((0,0))$. *Hence,* f *is continuous at* $(0,0)$.

It follows from the definition and Theorem 5.1.19 that continuity is transmitted to sums, products, and quotients when the ranges of our functions are subsets of the complex field. For completeness, the general result is stated in the following theorem.

Theorem 5.2.9 *If f and g are complex valued functions that are continuous on a metric space X, then $f+g$ and fg are continuous on X. Furthermore, $\dfrac{f}{g}$ is continuous on $X-\{p\in X : g(p)=0\}$.*

From Lemma 4.3.1, it follows immediately that functions from arbitrary metric spaces to Euclidean k-space are continuous if and only if they are continuous by coordinate. Furthermore, Theorem 5.1.23 tells us that continuity is transmitted to sums and inner products.

Theorem 5.2.10 *(a) Let $f_1, f_2, ..., f_k$ be real valued functions on a metric space X, and $\mathbf{F} : X \longrightarrow \mathbb{R}^k$ be defined by $\mathbf{F}(x) = (f_1(x), f_2(x), ..., f_k(x))$. Then $\mathbf{F}$ is continuous if and only if f_j is continuous for each j, $1 \le j \le k$.*

(b) If $\mathbf{f}$ and $\mathbf{g}$ are continuous functions from a metric space X into $\mathbb{R}^k$, then $\mathbf{f}+\mathbf{g}$ and $\mathbf{f}\bullet\mathbf{g}$ are continuous on X.

The other combination of functions that we wish to examine on arbitrary metric spaces is that of composition. If X, Y, and Z are metric spaces, $E\subset X$, $f : E\longrightarrow Y$, and $g : f(E)\longrightarrow Z$, then the composition of f and g, denoted by $g\circ f$, is defined by $g(f(x))$ for each $x\in E$. The following theorem tells us that continuity is transmitted through composition.

Theorem 5.2.11 *Suppose that X, Y, and Z are metric spaces, $E\subset X$, $f : E\longrightarrow Y$, and $g : f(E)\longrightarrow Z$. If f is continuous at $p\in E$ and g is continuous at $f(p)$, then the composition $g\circ f$ is continuous at $p\in E$.*

Space for scratch work.

Proof. Suppose that (X, d_X), (Y, d_Y), and (Z, d_Z) are metric spaces, $E\subset X$, $f : E\longrightarrow Y$, $g : f(E)\longrightarrow Z$, f is continuous at $p\in E$, and g is continuous at $f(p)$. Let $\varepsilon>0$ be given. Since g is continuous at $f(p)$, there exists a positive real number δ_1 such that $d_Z(g(y), g(f(p))) < \varepsilon$ for any $y \in f(E)$ such that $d_Y(y, f(p)) < \delta_1$. From f being continuous at $p \in E$ and δ_1 being a positive real number, we deduce the existence of another positive real number δ such that $x \in E$ and $d_X(x, p) < \delta$ implies that $d_Y(f(x), f(p)) < \delta_1$. Substituting $f(x)$ for y, we have that $x \in E$ and $d_X(x, p) < \delta$ implies that $d_Y(f(x), f(p)) < \delta_1$ which further implies that $d_Z(g(f(x)), g(f(p))) < \varepsilon$. That is, $d_Z((g \circ f)(x), (g \circ f)(p)) < \varepsilon$ for any $x \in E$ for which $d_X(x, p) < \delta$. Therefore, $g \circ f$ is continuous at p. ■

Remark 5.2.12 *The "with respect to a set" distinction can be an important one to note. For example, the function* $f(x) = \begin{cases} 1 & , \textit{ for } x \textit{ rational} \\ 0 & , \textit{ for } x \textit{ irrational} \end{cases}$ *is continuous with respect to the rationals and it is continuous with respect to the irrationals. However, it is not continuous on* $\mathbb{R}^1$.

5.2.1 A Characterization of Continuity

Because continuity is defined in terms of proximity, it can be helpful to rewrite the definition in terms of neighborhoods. Recall that, for (X, d_X), $p \in X$, and $\delta > 0$,

$$N_\delta(p) = \{x \in X : d_X(x, p) < \delta\}.$$

For a metric space (Y, d_Y), $f : X \longrightarrow Y$ and $\varepsilon > 0$,

$$N_\varepsilon(f(p)) = \{y \in Y : d_Y(y, f(p)) < \varepsilon\}.$$

Hence, for metric spaces (X, d_X) and (Y, d_Y), $E \subset X$, $f : E \longrightarrow Y$ and $p \in E$, f is continuous at p if and only if

$$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon) > 0)\left[f(N_\delta(p) \cap E) \subset N_\varepsilon(f(p))\right].$$

Because neighborhoods are used to define open sets, the neighborhood formulation for the definition of continuity of a function points us in the direction of the following theorem.

Theorem 5.2.13 (Open Set Characterization of Continuous Functions) *Let f be a mapping on a metric space (X, d_X) into a metric space (Y, d_Y). Then f is continuous on X if and only if for every open set V in Y, the set $f^{-1}(V)$ is open in X.*

Space for scratch work.

Excursion 5.2.14 *Fill in what is missing in order to complete the following proof of the theorem.*

Proof. Let f be a mapping from a metric space (X, d_X) into a metric space (Y, d_Y).

Suppose that f is continuous on X, V is an open set in Y, and $p_0 \in f^{-1}(V)$. Since V is open and $f(p_0) \in V$, we can choose $\varepsilon > 0$ such that $N_\varepsilon(f(p_0)) \subset V$ from which it follows that

$$\underbrace{\hspace{3cm}}_{(1)} \subset f^{-1}(V).$$

Because f is continuous at $p_0 \in X$, corresponding to $\varepsilon > 0$, there exists a $\delta > 0$, such that $f(N_\delta(p_0)) \subset N_\varepsilon(f(p_0))$ which implies that

$$N_\delta(p_0) \subset \underbrace{\hspace{3cm}}_{(2)}.$$

From the transitivity of subset, we concluded that $N_\delta(p_0) \subset f^{-1}(V)$. Hence, p_0 is an interior point of $f^{-1}(V)$. Since p_0 was arbitrary, we conclude that each $p \in f^{-1}(V)$ is an interior point. Therefore, $f^{-1}(V)$ is open.

To prove the converse, suppose that the inverse image of every open set in Y is open in X. Let p be an element in X and $\varepsilon > 0$ be given. Now the neighborhood $N_\varepsilon(f(p))$ is open in Y. Consequently, $\underbrace{\hspace{3cm}}_{(3)}$ is open in X.

Since p is an element of $\underset{(3)}{\underline{\hspace{5em}}}$, there exists a positive real number δ such that $N_\delta(p) \subset f^{-1}(N_\varepsilon(f(p)))$; i.e., $\underset{(4)}{\underline{\hspace{5em}}} \subset N_\varepsilon(f(p))$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{x \to p} f(x) = \underset{(5)}{\underline{\hspace{3em}}}$. Finally, because p was an arbitrary point in X, it follows that f $\underset{(6)}{\underline{\hspace{10em}}}$ as needed. ■

***Acceptable responses are: (1) $f^{-1}(N_\varepsilon(f(p_0))$, (2) $f^{-1}(N_\varepsilon(f(p_0)))$,
(3) $f^{-1}(N_\varepsilon(f(p)))$, (4) $f(N_\delta(p))$, (5) $f(p)$, (6) is continuous on X.***

Excursion 5.2.15 *Suppose that f is a mapping on a metric space (X, d_X) into a metric space (Y, d_Y) and $E \subset X$. Prove that $f^{-1}[E^c] = (f^{-1}[E])^c$.*

The following corollary follows immediately from the Open Set Characterization for Continuity, Excursion 5.2.15, and the fact that a set is closed if and only if its complement is open. Use the space provided after the statement to convince yourself of the truth of the given statement.

Corollary 5.2.16 *A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.*

Remark 5.2.17 *We have stated results in terms of open sets in the full metric space. We could also discuss functions restricted to subsets of metric spaces and then the characterization would be in terms of relative openness. Recall that given two sets X and Y and $f : X \longrightarrow Y$, the corresponding set induced functions satisfy the following properties for $C_j \subset X$ and $D_j \subset Y$, $j = 1, 2$:*

- $f^{-1}[D_1 \cap D_2] = f^{-1}[D_1] \cap f^{-1}[D_2]$,
- $f^{-1}[D_1 \cup D_2] = f^{-1}[D_1] \cup f^{-1}[D_2]$,
- $f[C_1 \cap C_2] \subset f[C_1] \cap f[C_2]$, *and*
- $f[C_1 \cup C_2] = f[C_1] \cup f[C_2]$

Because subsets being open to subsets of metric spaces in characterized by their realization as intersections with open subsets of the parent metric spaces, our neighborhoods characterization tells us that we loose nothing by looking at restrictions of given functions to the subsets that we wish to consider rather that stating things in terms of relative openness.

5.2.2 Continuity and Compactness

Theorem 5.2.18 *If f is a continuous function from a compact metric space X to a metric space Y, then $f(X)$ is compact.*

Excursion 5.2.19 *Fill in what is missing to complete the following proof of Theorem 5.2.18.*

Space for scratch work.

Proof. Suppose that f is a continuous function from a compact metric space X to a metric space Y and $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ is an open cover for $f(X)$. Then G_α is open in Y for each $\alpha \in \Delta$ and $\underset{(1)}{\underline{\hspace{3cm}}}$.

From the Open Set Characterization of $\underset{(2)}{\underline{\hspace{3cm}}}$ Functions, $f^{-1}(G_\alpha)$ is $\underset{(3)}{\underline{\hspace{3cm}}}$ for each $\alpha \in \Delta$.

Since $f : X \twoheadrightarrow f(X)$ and $f(X) \subset \bigcup_{\alpha \in \Delta} G_\alpha$, we have that

$$X = f^{-1}(f(X)) \subset f^{-1}\left(\bigcup_{\alpha \in \Delta} G_\alpha\right) = \underset{(4)}{\underline{\hspace{3cm}}}.$$

Hence, $\mathcal{F} = \left\{ f^{-1}\left(G_\alpha\right) : \alpha \in \Delta \right\}$ is an $\underset{(5)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}$ for X. Since X is $\underset{(6)}{\underline{\qquad\qquad\qquad}}$, there is a finite subcollection of $\mathcal{F}$, $\left\{ f^{-1}\left(G_{\alpha_j}\right) : j = 1, 2, ..., n \right\}$, that covers X; i.e.,

$$X \subset \bigcup_{j=1}^{n} f^{-1}\left(G_{\alpha_j}\right).$$

It follows that

$$f\left(X\right) \subset f\left(\bigcup_{j=1}^{n} f^{-1}\left(G_{\alpha_j}\right) \right) = \bigcup_{j=1}^{n} \underset{(7)}{\underline{\qquad\quad}} = \bigcup_{j=1}^{n} G_{\alpha_j}.$$

Therefore, $\left\{ G_{\alpha_j} : j = 1, 2, ..., n \right\}$ is a finite subcollection of $\mathcal{G}$ that covers $f\left(X\right)$. Since $\mathcal{G}$ was arbitrary, every $\underset{(8)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}$; i.e., $f\left(X\right)$ is $\underset{(9)}{\underline{\qquad\qquad\qquad}}$. ■

Remark 5.2.20 *Just to stress the point, in view of our definition of relative compactness the result just stated is also telling us that the continuous image of any compact subset of a metric space is a compact subset in the image.*

Definition 5.2.21 *For a set E, a function $\mathbf{f} : E \longrightarrow \mathbb{R}^k$ is said to be **bounded** if and only if*

$$(\exists M)\left(M \in \mathbb{R} \wedge (\forall x)\left(x \in E \Rightarrow \left|\mathbf{f}\left(x\right)\right| \leq M\right)\right).$$

When we add compactness to domain in the metric space, we get some nice analogs.

Theorem 5.2.22 (Boundedness Theorem) *Let A be a compact subset of a metric space (S, d) and suppose that $\mathbf{f} : A \longrightarrow \mathbb{R}^k$ is continuous. Then $f\left(A\right)$ is closed and bounded. In particular, f is bounded.*

Excursion 5.2.23 *Fill in the blanks to complete the following proof of the Boundedness Theorem.*

Proof. *By the* $\underset{(1)}{\underline{\hspace{6cm}}}$, *we know that compactness in* $\mathbb{R}^k$ *for any* $k \in \mathbb{J}$ *is equivalent to being closed and bounded. Hence, from Theorem 5.2.18, if* $f : A \longrightarrow \mathbb{R}^k$ *where* A *is a compact metric space, then* $f(A)$ *is compact. But* $f(A) \subset \underset{(2)}{\underline{\hspace{1.5cm}}}$ *and compact yields that* $f(A)$ *is* $\underset{(3)}{\underline{\hspace{6cm}}}$. *In particular,* $f(A)$ *is bounded as claimed in the Boundedness Theorem.* ■

Expected responses are: (1) Heine-Borel Theorem, (2) $\mathbb{R}^k$, and (3) closed and bounded.

Theorem 5.2.24 (Extreme Value Theorem) *Suppose that* f *is a continuous function from a compact subset* A *of a metric space* S *into* $\mathbb{R}^1$,

$$M = \sup_{p \in A} f(p) \qquad \text{and} \qquad m = \inf_{p \in A} f(p).$$

Then there exist points u *and* v *in* A *such that* $f(u) = M$ *and* $f(v) = m$.

Proof. From Theorem 5.2.18 and the Heine-Borel Theorem, $f(A) \subset \mathbb{R}$ and f continuous implies that $f(A)$ is closed and bounded. The Least Upper and Greatest Lower Bound Properties for the reals yields the existence of finite real numbers M and m such that $M = \sup_{p \in A} f(p)$ and $m = \inf_{p \in A} f(p)$. Since $f(A)$ is closed, by Theorem 3.3.26, $M \in f(A)$ and $m \in f(A)$. Hence, there exists u and v in A such that $f(u) = M$ and $f(v) = m$; i.e., $f(u) = \sup_{p \in A} f(p)$ and $f(v) = \inf_{p \in A} f(p)$. ■

Theorem 5.2.25 *Suppose that* f *is a continuous one-to-one mapping of a compact metric space* X *onto a metric space* Y. *Then the inverse mapping* f^{-1} *which is defined by* $f^{-1}(f(x)) = x$ *for all* $x \in X$ *is a continuous mapping that is a one-to-one correspondence from* Y *to* X.

Proof. *Suppose that* f *is a continuous one-to-one mapping of a compact metric space* X *onto a metric space* Y. *Because* f *is one-to-one, the inverse* f^{-1} *is a function from* $\operatorname{rng}(f) = Y$ *in* X. *From the Open Set Characterization of Continuous Functions, we know that* f^{-1} *is continuous in* Y *if* $f(U)$ *is open in* Y *for every* U *that is open in* X. *Suppose that* $U \subset X$ *is open. Then, by Theorem 3.3.37,* U^c *is compact as a closed subset of the compact metric space* X. *In view of Theorem*

5.2.18, $f(U^c)$ *is compact. Since every compact subset of a metric space is closed (Theorem 3.3.35), we conclude that* $f(U^c)$ *is closed. Because* f *is one-to-one,* $f(U^c) = f(X) - f(U)$*; then* f *onto yields that* $f(U^c) = Y - f(U) = f(U)^c$*. Therefore,* $f(U)^c$ *is closed which is equivalent to* $f(U)$ *being open. Since* U *was arbitrary, for every* U *open in* X*, we have that* $f(U)$ *is open in* Y*. Hence,* f^{-1} *is continuous in* Y*.* ■

5.2.3 Continuity and Connectedness

Theorem 5.2.26 *Suppose that* f *is a continuous mapping for a metric space* X *into a metric space* Y *and* $E \subset X$*. If* E *is a connected subset of* X*, then* $f(E)$ *is connected in* Y*.*

Excursion 5.2.27 *Fill in what is missing in order to complete the following proof of Theorem 5.2.26.*

Space for scratch work.

Proof. Suppose that f is a continuous mapping from a metric space X into a metric space Y and $E \subset X$ is such that $f(E)$ is not connected. Then we can let $f(E) = A \cup B$ where A and B are nonempty $\underset{(1)}{\underline{\hspace{3cm}}}$ subsets of Y; i.e., $A \neq \emptyset$, $B \neq \emptyset$ and $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Consider $G \underset{def}{=} E \cap f^{-1}(A)$ and $H \underset{def}{=} E \cap f^{-1}(B)$. Then neither G nor H is empty and

$$\begin{aligned} G \cup H &= \left(E \cap f^{-1}(A)\right) \cup \underset{(2)}{\underline{\hspace{3cm}}} \\ &= E \cap \left(\underset{(3)}{\underline{\hspace{3cm}}}\right) \\ &= E \cap f^{-1}(A \cup B) = \underset{(4)}{\underline{\hspace{3cm}}}. \end{aligned}$$

Because $A \subset \overline{A}$, $f^{-1}(A) \subset f^{-1}\left(\overline{A}\right)$. Since $G \subset f^{-1}(A)$, the transitivity of containment yields that $\underset{(5)}{\underline{\hspace{3cm}}}$. From the Corollary to the Open Set Characterization for Continuous Functions, $f^{-1}\left(\overline{A}\right)$ is $\underset{(6)}{\underline{\hspace{3cm}}}$.

It follows that $\overline{G} \subset f^{-1}\left(\overline{A}\right)$. From $\overline{G} \subset f^{-1}\left(\overline{A}\right)$ and
$H \subset f^{-1}(B)$, we have that

$$\overline{G} \cap H \subset f^{-1}\left(\overline{A}\right) \cap f^{-1}(B) = f^{-1}\left(\underset{(7)}{\underline{\hspace{3cm}}}\right) = f^{-1}\left(\underset{(8)}{\underline{\hspace{1cm}}}\right) = \underset{(9)}{\underline{\hspace{1cm}}}.$$

The same argument yields that $G \cap \overline{H} = \emptyset$.
From $E = G \cup H$, $G \neq \emptyset$, $H \neq \emptyset$ and ■
$\overline{G} \cap H = G \cap \overline{H} = \emptyset$, we conclude that E is
$\underset{(10)}{\underline{\hspace{3cm}}}$. Hence, for f a continuous mapping
from a metric space X into a metric space Y and $E \subset X$,
if $f(E)$ is not connected, then E is not connected.
According to the contrapositive, we conclude that, if
$\underset{(11)}{\underline{\hspace{3cm}}}$, then $\underset{(12)}{\underline{\hspace{3cm}}}$, as needed.

*******Acceptable responses are: (1) separated (2) $E \cap f^{-1}(B)$, (3) $f^{-1}(A) \cup f^{-1}(B)$, (4) E, (5) $G \subset f^{-1}\left(\overline{A}\right)$, (6) closed, (7) $\overline{A} \cap B$, (8) $\emptyset$, (9) $\emptyset$, (10) not connected, (11) E is connected, and (12) $f(E)$ is connected.*******

Theorem 5.2.28 *Suppose that f is a real-valued function on a metric space (X, d). If f is continuous on S, a nonempty connected subset of X, then the range of $f|_S$, denoted by $R(f|_S)$, is either an interval or a point.*

Theorem 5.2.29 (The Intermediate Value Theorem) *Let f be a continuous real-valued function on an interval $[a, b]$. If $f(a) < f(b)$ and if $c \in (f(a), f(b))$, then there exist a point $x \in (a, b)$ such that $f(x) = c$.*

Proof. Let $E = f([a, b])$. Because $[a, b]$ is an interval, from Theorem 3.3.60, we know that $[a, b]$ is connected. By Theorem 5.2.26, E is also connected as the continuous image of a connected set. Since $f(a)$ and $f(b)$ are in E, from Theorem 3.3.60, it follows that if c is a real number satisfying $f(a) < c < f(b)$, then c is in E. Hence, there exists x in $[a, b]$ such that $f(x) = c$. Since $f(a)$ is not equal to c and $f(b)$ is not equal to c, we conclude that x is in (a, b). Therefore, there exists x in (a, b) such that $f(x) = c$, as claimed. ■

5.3 Uniform Continuity

Our definition of continuity works from continuity at a point. Consequently, point dependency is tied to our $\delta - \varepsilon$ proofs of limits. For example, if we carried out a $\delta - \varepsilon$ proof that $f(x) = \dfrac{2x+1}{x-1}$ is continuous at $x = 2$, corresponding to $\varepsilon > 0$, taking $\delta = \min\left\{\dfrac{1}{2}, \dfrac{\varepsilon}{6}\right\}$ will work nicely to show that $\lim\limits_{x\to 2} \dfrac{2x+1}{x-1} = 5 = f(2)$; however, it would not work for showing continuity at $x = \dfrac{3}{2}$. On the other hand, corresponding to $\varepsilon > 0$, taking $\delta = \min\left\{\dfrac{1}{4}, \dfrac{\varepsilon}{24}\right\}$ will work nicely to show that $\lim\limits_{x\to\frac{3}{2}} \dfrac{2x+1}{x-1} = 8 = f\left(\dfrac{3}{2}\right)$. The point dependence of the work is just buried in the focus on the local behavior. The next concept demands a "niceness" that is global.

Definition 5.3.1 *Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \longrightarrow Y$ is* ***uniformly continuous*** *on X if and only if*

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\left[(\forall p)\,(\forall q)\,(p, q \in X \wedge d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon)\right].$$

Example 5.3.2 *The function $f(x) = x^2 : \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous on* $[1, 3]$.

For $\varepsilon > 0$ let $\delta = \dfrac{\varepsilon}{6}$. For $x_1, x_2 \in [1, 3]$, the triangular inequality yields that

$$|x_1 + x_2| \leq |x_1| + |x_2| \leq 6.$$

Hence, $x_1, x_2 \in [1, 3]$ and $|x_1 - x_2| < \delta$ implies that

$$|f(x_1) - f(x_2)| = \left|x_1^2 - x_2^2\right| = |x_1 - x_2|\,|x_1 + x_2| < \delta \cdot 6 = \varepsilon.$$

Since $\varepsilon > 0$ and $x_1, x_2 \in [1, 3]$ were arbitrary, we conclude that f is uniformly continuous on $[1, 3]$.

Example 5.3.3 *The function $f(x) = x^2 : \mathbb{R} \longrightarrow \mathbb{R}$ is not uniformly continuous on* $\mathbb{R}$.

We want to show that there exists a positive real number ε *such that corresponding to every positive real number* δ *we have (at least two points)* $x_1 = x_1(\delta)$ *and* $x_2 = x_2(\delta)$ *for which* $|x_1(\delta) - x_2(\delta)| < \delta$ *and* $|f(x_1) - f(x_2)| \geq \varepsilon$. *This statement is an exact translation of the negation of the definition. For the given function, we want to exploit the fact the as* x *increases* x^2 *increase at a rapid (not really uniform) rate.*

Take $\varepsilon = 1$. *For any positive real number* δ, *let* $x_1 = x_1(\delta) = \dfrac{\delta}{2} + \dfrac{1}{\delta}$ *and* $x_2 = x_2(\delta) = \dfrac{1}{\delta}$. *Then* x_1 *and* x_2 *are real numbers such that*

$$|x_1 - x_2| = \left|\left(\frac{\delta}{2} + \frac{1}{\delta}\right) - \left(\frac{1}{\delta}\right)\right| = \frac{\delta}{2} < \delta$$

while

$$|f(x_1) - f(x_2)| = \left|x_1^2 - x_2^2\right| = |x_1 - x_2|\,|x_1 + x_2| = \left(\frac{\delta}{2}\right)\left(\frac{\delta}{2} + \frac{2}{\delta}\right) = \frac{\delta^2}{4} + 1 \geq 1 = \varepsilon.$$

Hence, f *is not uniformly continuous on* $\mathbb{R}$.

Example 5.3.4 *For* $p \in \mathbb{R}^1$, *let* $f(p) = (2p + 1,\ p^2)$. *Then* $f : \mathbb{R}^1 \longrightarrow \mathbb{R}^2$ *is uniformly continuous on the closed interval* $[0, 2]$.

For $\varepsilon > 0$, *let* $\delta = \dfrac{\varepsilon}{2\sqrt{5}}$. *If* $p_1 \in [0, 2]$ *and* $p_2 \in [0, 2]$, *then*

$$4 + (p_1 + p_2)^2 = 4 + |p_1 + p_2|^2 \leq 4 + (|p_1| + |p_2|)^2 \leq 4 + (2 + 2)^2 = 20$$

and

$$\begin{aligned} d_{\mathbb{R}^2}(f(p_1), f(p_2)) &= \sqrt{((2p_1 + 1) - (2p_2 + 1))^2 + \left(p_1^2 - p_2^2\right)^2} \\ &= |p_1 - p_2|\sqrt{4 + (p_1 + p_2)^2} < \delta \cdot \sqrt{20} = \frac{\varepsilon}{2\sqrt{5}} \cdot \sqrt{20} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ *and* $p_1, p_2 \in [1, 2]$ *were arbitrary, we conclude that* f *is uniformly continuous on* $[0, 2]$.

Theorem 5.3.5 (Uniform Continuity Theorem) *If* f *is a continuous mapping from a compact metric space* X *to a metric space* Y, *then* f *is uniformly continuous on* X.

Excursion 5.3.6 *Fill in what is missing in order to complete the proof of the Uniform Continuity Theorem.*

Proof. Suppose f is a continuous mapping from a compact metric space (X, d_X) to a metric space (Y, d_Y) and that $\varepsilon > 0$ is given. Since f is continuous, for each $p \in X$, there exists a positive real number δ_p such that $q \in X \wedge d_X(q, p) < \delta_p \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$. Let $\mathcal{G} = \left\{ N_{\frac{1}{2}\delta_p}(p) : p \in X \right\}$. Since neighborhoods are open sets, we conclude that $\mathcal{G}$ is an $\underset{(1)}{\underline{\hspace{6cm}}}$. Since X is compact there exists a finite number of elements of $\mathcal{G}$ that covers X, say $p_1, p_2, ..., p_n$. Hence,

$$X \subset \bigcup_{j=1}^{n} N_{\frac{1}{2}\delta_{p_j}}\left(p_j\right).$$

Let $\delta \underset{def}{=} \frac{1}{2} \min_{1 \le j \le n} \left\{\delta_{p_j}\right\}$. Then, $\delta > 0$ and the minimum of a finite number of positive real numbers.

Suppose that $p, q \in X$ are such that $d(p, q) < \delta$. Because $p \in X$ and $X \subset \bigcup_{j=1}^{n} N_{\frac{1}{2}\delta_{p_j}}\left(p_j\right)$, there exists a positive integer k, $1 \le k \le n$, such that $\underset{(2)}{\underline{\hspace{6cm}}}$. Hence, $d(p, p_k) < \frac{1}{2}\delta_{p_k}$. From the triangular inequality

$$d_X(q, p_k) \le d_X(q, p) + d_X(p, p_k) < \delta + \underset{(3)}{\underline{\hspace{1cm}}} \le \delta_{p_k}.$$

Another application of the triangular inequality and the choices that were made for δ_p yield that

$$d_Y(f(p), f(q)) \le \underset{(4)}{\underline{\hspace{6cm}}} < \underset{(5)}{\underline{\hspace{3cm}}} = \varepsilon$$

■

***Acceptable fill ins are: (1) open cover for X (2) $p \in N_{\frac{1}{2}\delta_k}(p_k)$, (3) $\frac{1}{2}\delta_{p_k}$, (4) $d_Y(f(p), f(p_k)) + d_Y(f(p_k), f(q))$, $\frac{\varepsilon}{2} + \frac{\varepsilon}{2}$. ***

5.4 Discontinuities and Monotonic Functions

Given two metric spaces (X, d_X) and (Y, d_Y) and a function f from a subset A of X into Y. If $p \in X$ and f is not continuous at p, then we can conclude that f is not defined at p ($p \notin A = \operatorname{dom}(f)$), $\lim\limits_{\substack{x \to p \\ x \in A}} f(x)$ does not exist, and/or $p \in A$ and $\lim\limits_{\substack{x \to p \\ x \in A}} f(x)$ exists but $f(p) \neq \lim\limits_{\substack{x \to p \\ x \in A}} f(x)$; a point for which any of the three conditions occurs is called a point of discontinuity. In a general discussion of continuity of given functions, there is no need to discuss behavior at points that are not in the domain of the function; consequently, our consideration of points of discontinuity is restricted to behavior at points that are in a specified or implied domain. Furthermore, our discussion will be restricted to points of discontinuity for real-valued functions of a real-variable. This allows us to talk about one-sided limits, behavior on both sides of discontinuities and growth behavior.

Definition 5.4.1 *A function f is* ***discontinuous*** *at a point $c \in \operatorname{dom}(f)$ or* ***has a discontinuity at*** *c if and only if either $\lim\limits_{x \to c} f(x)$ doesn't exist or $\lim\limits_{x \to c} f(x)$ exists and is different from $f(c)$.*

Example 5.4.2 *The domain of $f(x) = \dfrac{|x|}{x}$ is $\mathbb{R} - \{0\}$. Consequently, f has no points of discontinuity on its domain.*

Example 5.4.3 *For the function* $f(x) = \begin{cases} \dfrac{|x|}{x} & , \text{ for } x \in \mathbb{R} - \{0\} \\ 1 & , \text{ for } x = 0 \end{cases},$

$\operatorname{dom}(f) = \mathbb{R}$ *and $x = 0$ is a point of discontinuity of f. To see that $\lim\limits_{x \to 0} f(x)$ does not exist, note that, for every positive real number δ,*

$$f\left(-\frac{\delta}{2}\right) = -1 \qquad \text{and} \qquad \left| f\left(-\frac{\delta}{2}\right) - f(0) \right| = 2.$$

Hence, if $\varepsilon = \dfrac{1}{2}$, then, for every positive real number δ, there exists $x \in \operatorname{dom}(f)$ such that $0 < |x| < \delta$ and $|f(x) - f(0)| \geq \varepsilon$. Therefore, f is not continuous at $x = 0$.

Example 5.4.4 *If* $g(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right) & , \quad \text{for } x \in \mathbb{R} - \{0\} \\ 0 & , \quad \text{for } x = 0 \end{cases}$*, then* g *has no discontinuities in* $\mathbb{R}$.

Excursion 5.4.5 *Graph the following function* f *and find*

$$A = \{x \in \operatorname{dom}(f) : f \text{ is continuous at } x\}$$

and $B = \{x \in \operatorname{dom}(f) : f \text{ is discontinuous at } x\}$.

$$f(x) = \begin{cases} \dfrac{4x+1}{1+x} & , \quad x \le \dfrac{1}{2} \\ 2 & , \quad \dfrac{1}{2} < x \le 1 \\ -2x+4 & , \quad 1 < x \le 3 \\ \lfloor x \rfloor + 2 & , \quad 3 < x \le 6 \\ \dfrac{14(x-10)}{x-14} & , \quad (6 < x < 14) \vee (14 < x) \end{cases}$$

Hopefully, your graph revealed that $A = \mathbb{R} - \{-1, 3, 4, 5, 6, 14\}$ and $B = \{3, 4, 5, 6\}$.

Definition 5.4.6 *Let f be a function that is defined on the segment (a, b). Then, for any point $x \in [a, b)$, the **right-hand limit** is denoted by $f(x+)$ and*

$$f(x+) = q \Leftrightarrow \left(\forall \{t_n\}_{n=1}^{\infty}\right) \left[\left(\{t_n\} \subset (x, b) \wedge \lim_{n\to\infty} t_n = x\right) \Rightarrow \lim_{n\to\infty} f(t_n) = q\right]$$

*and, for any $x \in (a, b]$, the **left-hand limit** is denoted by $f(x-)$ and*

$$f(x-) = q \Leftrightarrow \left(\forall \{t_n\}_{n=1}^{\infty}\right) \left[\left(\{t_n\} \subset (a, x) \wedge \lim_{n\to\infty} t_n = x\right) \Rightarrow \lim_{n\to\infty} f(t_n) = q\right].$$

Remark 5.4.7 *From the treatment of one-sided limits in frosh calculus courses, recall that $\lim_{t\to x^+} f(t) = q$ if and only if*

$$(\forall \varepsilon > 0)\,(\exists \delta = \delta(\varepsilon) > 0)\left[(\forall t)\,(t \in \operatorname{dom}(f) \wedge x < t < x + \delta \Rightarrow |f(t) - q| < \varepsilon)\right]$$

and $\lim_{t\to x^-} f(t) = q$ if and only if

$$(\forall \varepsilon > 0)\,(\exists \delta = \delta(\varepsilon) > 0)\left[(\forall t)\,(t \in \operatorname{dom}(f) \wedge x - \delta < t < x \Rightarrow |f(t) - q| < \varepsilon)\right].$$

The Sequences Characterization for Limits of Functions justifies that these definitions are equivalent to the definitions of $f(x+)$ and $f(x-)$, respectively.

Excursion 5.4.8 *Find $f(x+)$ and $f(x-)$ for every $x \in B$ where B is defined in Excursion 5.4.5.*

For this function, we have $f(3-) = -2$, $f(3+) = 5$; $f(4-) = 5$, $f(4+) = 6$; $f(5-) = 6$, $f(5+) = 7$; and $f(6-) = 7$, $f(6+) = 7$.

Remark 5.4.9 *For each point x where a function f is continuous, we must have $f(x+) = f(x-) = f(x)$.*

Definition 5.4.10 *Suppose the function f is defined on the segment (a, b) and discontinuous at $x \in (a, b)$. Then f has a* ***discontinuity of the first kind at*** *x or a* ***simple discontinuity at*** *x if and only if both $f(x+)$ and $f(x-)$ exist. Otherwise, the discontinuity is said to be a* ***discontinuity of the second kind****.*

Excursion 5.4.11 *Classify the discontinuities of the function f in Excursion 5.4.5.*

Remark 5.4.12 *The function* $F(x) = \begin{cases} f(x) & \text{, for } x \in R - \{-1, 14\} \\ 0 & \text{, for } x = -1 \vee x = 14 \end{cases}$ *where f is given in Excursion 5.4.5 has discontinuities of the second kind at $x = -1$ and $x = 14$.*

Excursion 5.4.13 *Discuss the continuity of each of the following.*

1. $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x + 2} & , \quad x < -2 \\ 2x - 1 & , \quad x \geq -2 \end{cases}$

2. $g(x) = \begin{cases} 2 & , \quad x \text{ rational} \\ 1 & , \quad x \text{ irrational} \end{cases}$

Your discussion of (1) combines considers some cases. For $-\infty < x < -2$, $\dfrac{x^2-x-6}{x+2}$ is continuous as the quotient of polynomials for which the denominator is not going to zero, while continuity of $2x-1$ for $x > -2$ follows from the limit of the sum theorem or because $2x-1$ is a polynomial; consequently, the only point in the domain of f that needs to be checked is $x=-2$. Since $f((-2)-) = \lim_{x\to -2^-} \dfrac{x^2-x-6}{x+2} = \lim_{x\to -2^-}(x-3) = -5$, $f((-2)+) = \lim_{x\to -2^-}(2x-1) = -5$, and $f(-2) = -5$, it follows that f is also continuous at $x=-2$. That the function given in (2) is not continuous anywhere follows from the density of the rationals and the irrationals; each point of discontinuity is a "discontinuity of the second kind."

Definition 5.4.14 *Let f be a real-valued function on a segment (a,b). Then f is said to be* ***monotonically increasing*** *on (a,b) if and only if*

$$(\forall x_1)(\forall x_2)\left[x_1, x_2 \in (a,b) \wedge x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)\right]$$

and f is said to be ***monotonically decreasing*** *on (a,b) if and only if*

$$(\forall x_1)(\forall x_2)\left[x_1, x_2 \in (a,b) \wedge x_1 > x_2 \Rightarrow f(x_1) \le f(x_2)\right]$$

The ***class of monotonic functions*** *is the set consisting of both the functions that are increasing and the functions that are decreasing.*

Excursion 5.4.15 *Classify the monotonicity of the function f that was defined in Excursion 5.4.5*

Based on the graph, we have that f is monotonically increasing in each of $(-\infty,-1)$, $(-1,1)$, and $(3,6)$; the function is monotonically decreasing in each of $\left(\frac{1}{2},3\right)$, $(6,14)$, and $(14,\infty)$. The section $\left(\frac{1}{2},1\right)$ is included in both statements because the function is constant there. As an alternative, we could have claimed that f is both monotonically increasing and monotonically decreasing in each of $\left(\frac{1}{2}, 1\right)$, $(3, 4)$, $(4, 5)$, and $(5, 6)$ and distinguished the other segments according to the property of being **strictl**y monotonically increasing and **strictly** monotonically decreasing.

Now we will show that monotonic functions do not have discontinuities of the second kind.

Theorem 5.4.16 *Suppose that the real-valued function f is monotonically increasing on a segment (a, b). Then, for every $x \in (a, b)$ both $f(x-)$ and $f(x+)$ exist,*

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$$

and

$$(\forall x)(\forall y)(a < x < y < b \Rightarrow f(x+) \leq f(y-)).$$

Excursion 5.4.17 *Fill in what is missing in order to complete the following proof of the theorem.*

Space for scratch work.

Proof. Suppose that f is monotonically increasing on the segment (a, b) and $x \in (a, b)$. Then, for every $t \in (a, b)$ such that $a < t < x$, $\underset{(1)}{\underline{\hspace{3cm}}}$. Hence, $B \underset{def}{=} \{f(t) : a < t < x\}$ is bounded above by $f(x)$. By the $\underset{(2)}{\underline{\hspace{6cm}}}$, the set B has a least upper bound; let $u = \sup(B)$. Now we want to show that $u = f(x-)$.

Let $\varepsilon > 0$ be given. Then $u = \sup(B)$ and $u - \varepsilon < u$ yields the existence of a $w \in B$ such that $\underset{(3)}{\underline{\hspace{6cm}}}$. From the definition of B, w is the image of a point in $\underset{(4)}{\underline{\hspace{3cm}}}$. Let $\delta > 0$

be such that $x - \delta \in (a, x)$ and $f(x - \delta) = w$. If $t \in (x - \delta, x)$, then

$$f(x - \delta) \le f(t) \qquad \mathit{and} \qquad \underset{(5)}{\underline{\qquad\qquad\qquad\qquad\qquad}}.$$

Since $u - \varepsilon < w$ and $f(x) \le u$, the transitivity of less than or equal to yields that

$$\underset{(6)}{\underline{\qquad\qquad\qquad}} < f(t) \qquad \mathit{and} \qquad f(t) \le \underset{(7)}{\underline{\qquad\qquad\qquad}}.$$

Because t was arbitrary, we conclude that

$$(\forall t)\,(x - \delta < t < x \Rightarrow u - \varepsilon < f(t) \le u)\,.$$

Finally, it follows from $\varepsilon > 0$ being arbitrary that

$$(\forall \varepsilon > 0)\left(\underset{(8)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}\right); \text{ i.e.,}$$

$$f(x-) = \lim_{t \to x^-} f(t) = u.$$

For every $t \in (a, b)$ such that $x < t < b$, we also have that $f(x) \le f(t)$ from which it follows that $C \underset{def}{=} \{f(t) : x < t < b\}$ is bounded $\underset{(9)}{\underline{\qquad\qquad\qquad}}$ by $f(x)$. From the greatest lower bound property of the reals, C has a greatest lower bound that we will denote by v.

Use the space provided to prove that $v = f(x+)$.

(10)

Next suppose that $x, y \in (a, b)$ are such that $x < y$. Because $f(x+) = \lim_{t \to x^+} f(t) = \inf\{f(t) : x < t < b\}$, $(x, y) \subset (x, b)$ and f is monotonically increasing, it follows that

$$\underset{(11)}{\underline{\qquad\qquad\qquad}} = \inf\{f(t) : x < t < y\}.$$

From our earlier discussion, ■
$f(y-) = \lim_{t \to y^-} f(t) = \underset{(12)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}$.
Now, $(x, y) \subset (a, y)$ yields that

$$f(y-) = \sup\{f(t) : x < t < y\}.$$

Therefore, $\underset{(13)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}$ as claimed.

The expected responses are: (1) $f(t) \leq f(x)$, (2) least upper bound property, (3) $u - \varepsilon < w < u$, (4) (a, x), (5) $f(t) \leq f(x)$, (6) $u - \varepsilon$, (7) u, (8) $(\exists\delta > 0)\left[(\forall t)\,(x - \delta < t < x \Rightarrow u - \varepsilon < f(t) < u)\right]$, (9) below, (10) Let $\varepsilon > 0$ be given. Then $v = \inf C$ implies that there exists $w \in C$ such that $v < w < v + \varepsilon$. Since $w \in C$, w is the image of some point in (x, b). Let $\delta > 0$ be such that $x + \delta \in (x, b)$ and $f(x + \delta) = w$. Now suppose $t \in (x, x + \delta)$. Then $f(x) \leq f(t)$ and $f(t) \leq f(x + \delta) = w$. Since $v \leq f(x)$ and $w < v + \varepsilon$, it follows that $v \leq f(t)$ and $f(t) < v + \varepsilon$. Thus, $(\exists\delta > 0)\left[(\forall t)\,(x < t < x + \delta \Rightarrow v < f(t) \leq v + \varepsilon)\right]$. Because $\varepsilon > 0$ was arbitrary, we conclude that $v = \lim_{t \to x^+} f(t) = f(x+)$., (11) $f(x+)$, (12) $\sup\{f(t) : a < t < y\}$ and (13) $f(x+) \leq f(y-)$.

Corollary 5.4.18 *Monotonic functions have no discontinuities of the second kind.*

The nature of discontinuities of functions that are monotonic on segments allows us to identify points of discontinuity with rationals in such a way to give us a limit on the number of them.

Theorem 5.4.19 *If f is monotonic on the segment (a, b), then*

$$\{x \in (a, b) : f \text{ is discontinuous at } x\}$$

is at most countable.

Excursion 5.4.20 *Fill in what is missing in order to complete the following proof of Theorem 5.4.19.*

Proof. *Without loss of generality, we assume that f is a function that is monotonically increasing in the segment $(a, b) \underset{def}{=} I$. If f is continuous in I, then f has no points of discontinuity there and we are done. Suppose that f is not continuous on I and let $D = \{w \in I : f \text{ is not continuous at } w\}$.*

From our assumption $D \neq \emptyset$ and we can suppose that $\zeta \in D$. Then $\zeta \in \operatorname{dom}(f)$,

$$(\forall x)\,(x \in I \wedge x < \zeta \Rightarrow f(x) \leq f(\zeta))$$

and

$$(\forall x)\left(x \in I \wedge \zeta < x \Rightarrow \underset{(1)}{\underline{\hspace{3cm}}}\right).$$

From Theorem 5.4.16, $f(\zeta-)$ and $f(\zeta+)$ exist; furthermore,

$$f(\zeta-) = \sup\{f(x) : x < \zeta\},\ f(\zeta+) = \underset{(2)}{\underline{\hspace{6cm}}}.$$

and $f(\zeta-) \leq f(\zeta+)$. Since ζ is a discontinuity for f, $f(\zeta-) \underset{(3)}{\underline{\hspace{1cm}}} f(\zeta+)$. From the Density of the Rationals, it follows that there exists a rational r_ζ such that $f(\zeta-) < r_\zeta < f(\zeta+)$. Let $I_{r_\zeta} = (f(\zeta-), f(\zeta+))$. If $D - \{\zeta\} = \emptyset$, then $|D| = 1$ and we are done. If $D - \{\zeta\} \neq \emptyset$ then we can choose another $\xi \in D$ such that $\xi \neq \zeta$. Without loss of generality suppose that $\xi \in D$ is such that $\zeta < \xi$. Since ζ was an arbitrary point in the discussion just completed, we know that there exists a rational r_ξ, $r_\xi \neq r_\zeta$, such that $\underset{(4)}{\underline{\hspace{6cm}}}$ and we can let $I_{r_\xi} = (f(\xi-), f(\xi+))$. Since $\zeta < \xi$, it follows from Theorem 5.4.16 that $\underset{(5)}{\underline{\hspace{2cm}}} \leq f(\xi-)$. Thus, $I_{r_\zeta} \cap I_{r_\xi} = \underset{(6)}{\underline{\hspace{0.7cm}}}$.

Now, let $\mathcal{G} = \left\{I_{r_\gamma} : \gamma \in D\right\}$ and $H : D \to \mathcal{G}$ be defined by $H(w) = I_{r_w}$. Now we claim that H is a one-to-one correspondence. To see that H is $\underline{\hspace{6cm}}$, suppose that $w_1, w_2 \in D$ and $H(w_1) = H(w_2)$.

Then

(7)

To see that H *is onto, note that by definition* $H(D) \subset \mathcal{G}$ *and suppose that* $A \in \mathcal{G}$. *Then*

(8)

Finally, $H : D \overset{1-1}{\twoheadrightarrow} \mathcal{G}$ *yields that* $D \sim \mathcal{G}$. *Since* $r_\gamma \in \mathbb{Q}$ *for each* $I_{r_\gamma} \in \mathcal{G}$, *we have that* $|\mathcal{G}| \leq |\mathbb{Q}| = \aleph_0$. *Therefore,* $|D| \leq \aleph_0$*; i.e., D is* ________________________.
(9)

■

***Acceptable responses are : (1) $f(\zeta) \leq f(x)$, (2) $\inf\{f(x) : \zeta < x\}$, (3) $<$, (4) $f(\xi-) < r_\xi < f(\xi+)$, (5) $f(\zeta+)$, (6) $\emptyset$, (7) $I_{r_{w_1}} = I_{r_{w_2}}$. From the Trichotomy Law, we know that one and only one of $w_1 < w_2$, $w_1 = w_2$, or $w_2 < w_1$ holds. Since either $w_1 < w_2$ or $w_2 < w_1$ implies that $I_{r_{w_1}} \cap I_{r_{w_2}} = \emptyset$, we conclude that $w_1 = w_2$. Since w_1 and w_2 were arbitrary,
$(\forall w_1)(\forall w_2)[H(w_1) = H(w_2) \Rightarrow w_1 = w_2]$; i.e., H is one-to-one., (8) there exists $r \in \mathbb{Q}$ such that $A = I_r$ and $r \in (f(\lambda-), f(\lambda+))$ for some $\lambda \in D$. It follows that $H(\lambda) = A$ or $A \in H(D)$. Since A was an arbitrary element of $\mathcal{G}$, we have that $(\forall A)[A \in \mathcal{G} \Rightarrow A \in H(D)]$; i.e., $\mathcal{G} \subset H(D)$. From $H(D) \subset \mathcal{G}$ and $\mathcal{G} \subset H(D)$, we conclude that $\mathcal{G} = H(D)$. Hence, H is onto., and (9) at most countable.***

Remark 5.4.21 *The level of detail given in Excursion 5.4.20 was more that was needed in order to offer a well presented argument. Upon establishing the ability to associate an interval* I_{r_ζ} *with each* $\zeta \in D$ *that is labelled with a rational and justifying that the set of such intervals is pairwise disjoint, you can simply assert that you have established a one-to-one correspondence with a subset of the rationals and the set of discontinuities from which it follows that the set of discontinuities is at most countable. I chose the higher level of detail–which is also acceptable–in* *order to make it clearer where material prerequisite for this course is a part of the foundation on which we are building. For a really concise presentation of a proof of Theorem 5.4.19, see pages 96-97 of our text.*

Remark 5.4.22 *On page 97 of our text, it is noted by the author that the discontinuities of a monotonic function need not be isolated. In fact, given any countable subset E of a segment (a, b), he constructs a function f that is monotonic on (a, b) with E as set of all discontinuities of f in (a, b). More consideration of the example is requested in our exercises.*

5.4.1 Limits of Functions in the Extended Real Number System

Recall the various forms of definitions for limits of real valued functions in relationship to infinity:
Suppose that f is a real valued function on $\mathbb{R}$, c is a real number, and L real number, then

- $\lim_{x\to+\infty} f(x) = L \Leftrightarrow (\forall\varepsilon > 0)(\exists K > 0)(x > K \Rightarrow |f(x) - L| < \varepsilon)$
  $$\Leftrightarrow (\forall\varepsilon > 0)(\exists K > 0)(x > K \Rightarrow f(x) \in N_\varepsilon(L))$$
- $\lim_{x\to-\infty} f(x) = L \Leftrightarrow (\forall\varepsilon > 0)(\exists K > 0)(x < -K \Rightarrow |f(x) - L| < \varepsilon)$
  $$\Leftrightarrow (\forall\varepsilon > 0)(\exists K > 0)(x < -K \Rightarrow f(x) \in N_\varepsilon(L))$$
- $\lim_{x\to c} f(x) = +\infty \Leftrightarrow (\forall M \in \mathbb{R})(\exists\delta > 0)(0 < |x - c| < \delta \Rightarrow f(x) > M)$
  $$\Leftrightarrow (\forall M \in \mathbb{R})(\exists\delta > 0)\left(x \in N_\delta^d(c) \Rightarrow f(x) > M\right)$$
  where $N_\delta^d(c)$ denotes the deleted neighborhood of c, $N_\delta(c) - \{c\}$.
- $\lim_{x\to c} f(x) = -\infty \Leftrightarrow (\forall M \in \mathbb{R})(\exists\delta > 0)(0 < |x - c| < \delta \Rightarrow f(x) < M)$
  $$\Leftrightarrow (\forall M \in \mathbb{R})(\exists\delta > 0)\left(x \in N_\delta^d(c) \Rightarrow f(x) < M\right)$$

Based of the four that are given, complete each of the following.

- $\lim_{x\to+\infty} f(x) = +\infty \Leftrightarrow$__________________________________

- $\lim_{x \to +\infty} f(x) = -\infty \Leftrightarrow$____________________________________

- $\lim_{x \to -\infty} f(x) = +\infty \Leftrightarrow$____________________________________

- $\lim_{x \to -\infty} f(x) = -\infty \Leftrightarrow$____________________________________

Hopefully, the neighborhood formulation and the pattern of the various statements suggests that we could pull things together if we had comparable descriptions for neighborhoods of $+\infty$ and $-\infty$.

Definition 5.4.23 *For any positive real number K,*

$$N_K(\infty) = \{x \in \mathbb{R} \cup \{+\infty, -\infty\} : x > K\}$$

and $N_K(-\infty) = \{x \in \mathbb{R} \cup \{+\infty, -\infty\} : x < K\}$ are neighborhoods of $+\infty$ and $-\infty$, respectively.

With this notation we can consolidate the above definitions.

Definition 5.4.24 *Let f be a real valued function defined on $\mathbb{R}$. Then for A and c in the extended real number system, $\lim_{x \to c} f(x) = A$ if and only if for every neighborhood of A, $N(A)$ there exists a deleted neighborhood of c, $N^{*d}(c)$, such that $x \in N^{*d}(c)$ implies that $f(x) \in N(A)$. When specification is needed this will be referred to a the* ***limit of a function in the extended real number system****.*

Hopefully, the motivation that led us to this definition is enough to justify the claim that this definition agrees with the definition of $\lim_{x \to c} f(x) = A$ when c and A are real. Because the definition is the natural generalization and our proofs for the properties of limits of function built on information concerning neighborhoods, we note that we can establish some of the results with only minor modification in the proofs that have gone before. We will simply state analogs.

Theorem 5.4.25 *Let* f *be a real-valued function that is defined on a set* $E \subset \mathbb{R}$ *and suppose that* $\lim_{t\to c} f(t) = A$ *and* $\lim_{t\to c} f(t) = C$ *for* c, A, *and* C *in the extended real number system. Then* $A = C$.

Theorem 5.4.26 *Let* f *and* g *be real-valued functions that are defined on a set* $E \subset \mathbb{R}$ *and suppose that* $\lim_{t\to c} f(t) = A$ *and* $\lim_{t\to c} g(t) = B$ *for* c, A, *and* B *in the extended real number system. Then*

1. $\lim_{t\to c}(f+g)(t) = A + B$,
2. $\lim_{t\to c}(fg)(t) = AB$, *and*
3. $\lim_{t\to c}\left(\frac{f}{g}\right)(t) = \frac{A}{B}$

whenever the right hand side of the equation is defined.

Remark 5.4.27 *Theorem 5.4.26 is not applicable when the algebraic manipulations lead to the need to consider any of the expressions* $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, *or* $\frac{A}{0}$ *because none of these symbols are defined.*

The theorems in this section have no impact on the process that you use in order to find limits of real functions as x goes to infinity. At this point in the coverage of material, given a specific function, we find the limit as x goes to infinity by using simple algebraic manipulations that allow us to apply our theorems for algebraic combinations of functions having finite limits. We close this chapter with two examples that are intended as memory refreshers.

Example 5.4.28 *Find* $\lim_{x\to\infty} \frac{\left(x^2 - 3x^3 + 5\right) + i\left(x^3 + x\sin x\right)}{4x^3 - 7}$.

Since the given function is the quotient of two functions that go to infinity as x *goes to infinity, we factor in order to transform the given in to the quotient of functions that will have finite limits. In particular, we want to make use of the fact that, for any* $p \in J$, $\lim_{x\to\infty} \frac{1}{x^p} = 0$. *From*

$$\lim_{x\to\infty} \frac{\left(x^2 - 3x^3 + 5\right) + i\left(x^3 + x\sin x\right)}{4x^3 - 7} = \lim_{x\to\infty} \frac{\left(\frac{1}{x} - 3 + \frac{5}{x^3}\right) + i\left(1 + \frac{\sin x}{x^2}\right)}{\left(4 - \frac{7}{x^3}\right)},$$

The limit of the quotient and limit of the sum theorem yields that

$$\lim_{x\to\infty}\frac{\left(x^2-3x^3+5\right)+i\left(x^3+x\sin x\right)}{4x^3-7}=\frac{(0-3+0)+i\,(1+0)}{4-0}=\frac{-3+i}{4}.$$

Example 5.4.29 *Find* $\lim_{x\to-\infty}\left(\sqrt{2x^2+x+2}-\sqrt{2x^2-x-1}\right)$.

In its current form, it looks like the function is tending to $\infty-\infty$ *which is undefined. In this case, we will try "unrationalizing" the expression in order to get a quotient at will allow some elementary algebraic manipulations. Note that*

$$\begin{aligned}&\left(\sqrt{2x^2+x+2}-\sqrt{2x^2-x-1}\right)\\ &=\quad\frac{\left(\sqrt{2x^2+x+2}-\sqrt{2x^2-x-1}\right)\left(\sqrt{2x^2+x+2}+\sqrt{2x^2-x-1}\right)}{\left(\sqrt{2x^2+x+2}+\sqrt{2x^2-x-1}\right)}\\ &=\quad\frac{\left(2x^2+x+2\right)-\left(2x^2-x-1\right)}{\left(\sqrt{2x^2+x+2}+\sqrt{2x^2-x-1}\right)}\\ &=\quad\frac{2x+3}{\left(\sqrt{2x^2+x+2}+\sqrt{2x^2-x-1}\right)}.\end{aligned}$$

Furthermore, for $x<0$, $\sqrt{x^2}=|x|=-x$. *Hence,*

$$\begin{aligned}&\lim_{x\to-\infty}\left(\sqrt{2x^2+x+2}-\sqrt{2x^2-x-1}\right)\\ &=\quad\lim_{x\to-\infty}\frac{2x+3}{\left(\sqrt{2x^2+x+2}+\sqrt{2x^2-x-1}\right)}\\ &=\quad\lim_{x\to-\infty}\frac{2x+3}{\sqrt{x^2}\left(\sqrt{2+\frac{1}{x}+\frac{2}{x^2}}+\sqrt{2-\frac{1}{x}-\frac{1}{x^2}}\right)}\\ &=\quad\lim_{x\to-\infty}(-1)\frac{2+\frac{3}{x}}{\left(\sqrt{2+\frac{1}{x}+\frac{2}{x^2}}+\sqrt{2-\frac{1}{x}-\frac{1}{x^2}}\right)}\\ &=\quad(-1)\left(\frac{2}{\sqrt{2}+\sqrt{2}}\right)=\frac{-1}{\sqrt{2}}=\frac{-\sqrt{2}}{2}.\end{aligned}$$

5.5 Problem Set E

1. For each of the following real-valued functions of a real variable give a well-written $\delta - \varepsilon$ proof of the claim.

   (a) $\lim\limits_{x \to 2} \left(3x^2 - 2x + 1\right) = 9$

   (b) $\lim\limits_{x \to -1} 8x^2 = 8$

   (c) $\lim\limits_{x \to 16} \sqrt{x} = 4$

   (d) $\lim\limits_{x \to 1} \dfrac{3}{x-2} = -3$

   (e) $\lim\limits_{x \to 3} \dfrac{x+4}{2x-5} = 7.$

2. For each of the following real-valued functions of a real variable find the implicit domain and range.

   (a) $f(x) = \dfrac{\sin x}{x^2 - 1}$

   (b) $f(x) = \sqrt{2x+1}$

   (c) $f(x) = \dfrac{x}{x^2 + 5x + 6}$

3. Let $f(x) = \begin{cases} \dfrac{-3}{x+3} & , \quad x < 0 \\ \dfrac{|x-2|}{x-2} & , \quad 0 \le x < 2 \wedge x > 2 \\ 1 & , \quad x = 2 \end{cases}$

   (a) Sketch a graph for f.

   (b) Determine where the function f is continuous.

4. Let $f(x) = \begin{cases} \left|x^2 - 5x - 6\right| & \text{, for } \left|x - \frac{7}{2}\right| \geq \frac{5}{2} \\ \\ \sqrt{36 - 6x} & \text{, for } \left|x - \frac{7}{2}\right| < \frac{5}{2} \end{cases}$ and

   $g(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{, for } x \neq -1 \\ \\ 3 & \text{, for } x = -1 \end{cases}$.

   (a) Discuss the continuity of f at $x = 1$.

   (b) Discuss the continuity of $(fg)(x) = f(x)g(x)$ at $x = -1$.

5. For $f : \mathbb{C} \longrightarrow \mathbb{R}$ given by $f(z) = |z|$ give a $\delta - \varepsilon$ proof that $\lim_{z \to (1+i)} f(z) = \sqrt{2}$.

6. When it exists, find

   (a) $\lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2}, \sqrt{3x^2 + 2} \right)$

   (b) $\lim_{x \to 1} \left(\frac{x - 1}{x^2 + 3x - 4}, \sqrt{x^4 + 5}, \frac{|x - 1|}{x - 1} \right)$

7. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and suppose that $\lim_{x \to a} f(x) = L > 0$. Prove that

$$\lim_{x \to a} \sqrt{f(x)} = \sqrt{L}.$$

8. Using only appropriate definitions and elementary bounding processes, prove that if g is a real-valued function on $\mathbb{R}$ such that $\lim_{x \to a} g(x) = M \neq 0$, then $\lim_{x \to a} \frac{1}{[g(x)]^2} = \frac{1}{M^2}$.

9. Suppose that A is a subset of a metric space (S, d), $f : A \longrightarrow \mathbb{R}^1$, and $g : A \longrightarrow \mathbb{R}^1$. Prove each of the following.

   (a) If c is a real number and $f(p) = c$ for all $p \in A$, then, for any limit point p_0 of A, we have that $\lim_{\substack{p \to p_0 \\ p \in A}} f(p) = c$.

(b) If $f(p) = g(p)$ for all $p \in A - \{p_0\}$ were p_0 is a limit point of A and $\lim\limits_{\substack{p \to p_0 \\ p \in A}} f(p) = L$, then $\lim\limits_{\substack{p \to p_0 \\ p \in A}} g(p) = L$.

(c) If $f(p) \leq g(p)$ for all $p \in A$, $\lim\limits_{\substack{p \to p_0 \\ p \in A}} f(p) = L$ and $\lim\limits_{\substack{p \to p_0 \\ p \in A}} g(p) = M$, then $L \leq M$.

10. For each of the following functions on $\mathbb{R}^2$, determine whether or not the given function is continuous at $(0,0)$. Use $\delta - \varepsilon$ proofs to justify continuity or show lack of continuity by justifying that the needed limit does not exist.

(a) $$f((x,y)) = \begin{cases} \dfrac{xy^2}{\left(x^2+y^2\right)^2} & \text{, for} \quad (x,y) \neq (0,0) \\ 0 & \text{, for} \quad x = y = 0 \end{cases}$$

(b) $$f((x,y)) = \begin{cases} \dfrac{x^3y^3}{\left(x^2+y^2\right)^2} & \text{, for} \quad (x,y) \neq (0,0) \\ 0 & \text{, for} \quad x = y = 0 \end{cases}$$

(c) $$f((x,y)) = \begin{cases} \dfrac{x^2y^4}{\left(x^2+y^4\right)^2} & \text{, for} \quad (x,y) \neq (0,0) \\ 0 & \text{, for} \quad x = y = 0 \end{cases}$$

11. Discuss the uniform continuity of each of the following on the indicated set.

(a) $f(x) = \dfrac{x^2+1}{2x+3}$ in the interval $[4,9]$.

(b) $f(x) = x^3$ in $[1,\infty)$.

12. For $a < b$, let $\mathcal{C}([a,b])$ denote the set of all real valued functions that are continuous on the interval $[a,b]$. Prove that $d(f,g) = \max\limits_{a \leq x \leq b} |f(x) - g(x)|$ is a metric on $\mathcal{C}([a,b])$.

13. Correctly formulate the monotonically decreasing analog for Theorem 5.4.16 and prove it.

14. Suppose that f is monotonically increasing on a segment $I = (a, b)$ and that $(\exists M)\left[M \in \mathbb{R} \wedge (\forall x)\,(x \in I \Rightarrow f(x) \leq M)\right]$. Prove that there exists a real number C such that $C \leq M$ and $f(b-) = C$.

15. A function f defined on an interval $I = [a, b]$ is called strictly increasing on I if and only if $f(x_1) > f(x_2)$ whenever $x_1 > x_2$ for $x_1, x_2 \in I$. Furthermore, a function f is said to have the intermediate value properly in I if and only if for each c between $f(a)$ and $f(b)$ there is an $x_0 \in I$ such that $f(x_0) = c$. Prove that a function f that is strictly increasing and has the intermediate value property on an interval $I = [a, b]$ is continuous on (a, b).

16. Give an example of a real-valued function f that is continuous and bounded on $[0, \infty)$ while not satisfying the Extreme Value Theorem.

17. Suppose that f is uniformly continuous on the intervals I_1 and I_2. Prove that f is uniformly continuous on $S = I_1 \cup I_2$.

18. Suppose that a real-valued function f is continuous on I° where $I = [a, b]$. If $f(a+)$ and $f(b-)$ exist, show that the function

$$f_0(x) = \begin{cases} f(a+) & \text{, for} \quad x = a \\ f(x) & \text{, for} \quad a < x < b \\ f(b-) & \text{, for} \quad x = b \end{cases}$$

is uniformly continuous on I.

19. If a real valued function f is uniformly continuous on the half open interval $(0, 1]$, is it true that f is bounded there. Carefully justify the position taken.

Chapter 6

Differentiation: Our First View

We are now ready to reflect on a particular application of limits of functions; namely, the derivative of a function. This view will focus on the derivative of real-valued functions on subsets of $\mathbb{R}^1$. Looking at derivatives of functions in $\mathbb{R}^k$ requires a different enough perspective to necessitate separate treatment; this is done with Chapter 9 of our text. Except for the last section, our discussion is restricted to aspects of differential calculus of one variable. You should have seen most of the results in your first exposure to calculus–MAT21A on this campus. However, some of the results proved in this chapter were only stated when you first saw them and some of the results are more general than the earlier versions that you might have seen. The good news is that the presentation here isn't dependent on previous exposure to the topic; on the other hand, reflecting back on prior work that you did with the derivative can enhance your understanding and foster a deeper level of appreciation.

6.1 The Derivative

Definition 6.1.1 *A real-valued function f on a subset Ω of $\mathbb{R}$ is* ***differentiable at a point*** $\zeta \in \Omega$ *if and only if f is defined in an open interval containing ζ and*

$$\lim_{w \to \zeta} \frac{f(w) - f(\zeta)}{w - \zeta} \tag{6.1}$$

exists. The value of the limit is denoted by $f'(\zeta)$. The function is said to be differentiable on Ω if and only if it is differentiable at each $\zeta \in \Omega$.

Remark 6.1.2 *For a function f and a fixed point ζ, the expression*

$$\phi(w) = \frac{f(w) - f(\zeta)}{w - \zeta}$$

is one form of what is often referred to as a "difference quotient". Sometimes it is written as

$$\frac{\Delta f}{\Delta w}$$

where the Greek letter Δ is being offered as a reminder that difference starts with a "d". It is the latter form that motivates use of the notation $\dfrac{df}{dw}$ for the first derivative of f as a function of w. Other commonly used notations are D_w and D_1; these only become useful alternatives when we explore functions over several real variables.

There is an alternative form of (6.1) that is often more useful in terms of computation and formatting of proofs. Namely, if we let $w = \zeta + h$, (6.1) can be written as

$$\lim_{h \to 0} \frac{f(\zeta + h) - f(\zeta)}{h}. \tag{6.2}$$

Remark 6.1.3 *With the form given in (6.2), the difference quotient can be abbreviated as $\dfrac{\Delta f}{h}$.*

Definition 6.1.4 *A real-valued function f on a subset Ω of $\mathbb{R}$ is* ***right-hand differentiable at a point*** *$\zeta \in \Omega$ if and only if f is defined in a half open interval in the form $[\zeta, \zeta + \delta)$ for some $\delta > 0$ and the one-sided derivative from the right, denoted by $D^{+} f(\zeta)$,*

$$\lim_{h \to 0^{+}} \frac{f(\zeta + h) - f(\zeta)}{h}$$

exists; the function f is ***left-hand differentiable at a point*** *$\zeta \in \Omega$ if and only if f is defined in a half open interval in the form $(\zeta - \delta, \zeta]$ for some $\delta > 0$ and the one-sided derivative from the left, denoted by $D^{-} f(\zeta)$,*

$$\lim_{h \to 0^{-}} \frac{f(\zeta + h) - f(\zeta)}{h}$$

exists.

Definition 6.1.5 *A real-valued function f is* ***differentiable on a closed interval*** $[a, b]$ *if and only if f is differentiable in (a, b), right-hand differentiable at $x = a$ and left-hand differentiable at $x = b$.*

Example 6.1.6 *Use the definition to prove that* $f(x) = \dfrac{x+2}{x-1}$ *is differentiable at* $x = 2$.

Note that f is defined in the open interval $(1, 3)$ which contains $w = 2$. Furthermore,

$$\lim_{w\to 2} \frac{f(w) - f(2)}{w-2} = \lim_{w\to 2} \frac{\left(\dfrac{w+2}{w-1}\right) - \dfrac{4}{1}}{w-2} = \lim_{w\to 2} \frac{-3(w-2)}{w-2} = \lim_{w\to 2} (-3) = -3.$$

Hence, f is differentiable at $w = 2$ and $f'(2) = -3$.

Example 6.1.7 *Use the definition to prove that $g(x) = |x-2|$ is not differentiable at $x = 2$.*

Since $\operatorname{dom}(g) = \mathbb{R}$, *the function g is defined in any open interval that contains $x = 2$. Hence, g is differentiable at $x = 2$ if and only if*

$$\lim_{h\to 0} \frac{g(2+h) - g(2)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

exists. Let $\phi(h) = \dfrac{|h|}{h}$ *for $h \neq 0$. Note that*

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1 \qquad and \qquad \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1.$$

Thus, $\phi(0+) \neq \phi(0-)$ from which we conclude that $\lim_{h\to 0} \phi(h)$ does not exist. Therefore, g is not differentiable at $x = 2$.

Remark 6.1.8 *Because the function g given in Example 6.1.7 is left-hand differentiable at $x = 2$ and right-hand differentiable at $x = 2$, we have that g is differentiable in each of $(-\infty, 2]$ and $[2, \infty)$.*

Example 6.1.9 *Discuss the differentiability of each of the following at $x = 0$.*

1. $G(x) = \begin{cases} x \sin \dfrac{1}{x} & , \textit{for} \quad x \neq 0 \\ 0 & , \textit{for} \quad x = 0 \end{cases}$

2. $F(x) = \begin{cases} x^3 \sin \dfrac{1}{x} & , \textit{for} \quad x \neq 0 \\ 0 & , \textit{for} \quad x = 0 \end{cases}$

First of all, notice that, though the directions did not specify appeal to the definition, making use of the definition is the only viable option because of the way the function is defined. Discussing the differentiability of functions that are defined "in pieces" requires consideration of the pieces. On segments where the functions are realized as simple algebraic combinations of nice functions, the functions can be declared as differentiable based on noting the appropriate nice properties. If the function is defined one way at a point and a different way to the left and/or right, then appeal to the difference quotient is mandated.

For (1), we note that G is defined for all reals, consequently, it is defined in every interval that contains 0*. Thus, G is differentiable at* 0 *if and only if*

$$\lim_{h \to 0} \frac{G(0+h) - G(0)}{h} = \lim_{h \to 0} \frac{h \sin \frac{1}{h} - 0}{h} = \lim_{h \to 0} \left(\sin \frac{1}{h} \right)$$

exists. For $h \neq 0$*, let* $\phi(h) = \sin \frac{1}{h}$*. For each* $n \in \mathbb{J}$*, let* $p_n = \dfrac{2}{\pi(2n-1)}$*. Now,* $\{p_n\}_{n=1}^{\infty}$ *converges to* 0 *as n approaches infinity; but* $\{\phi(p_n)\}_{n=1}^{\infty} = \left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$ *diverges. From the Sequences Characterization for Limits of Functions (Theorem 5.1.15), we conclude that* $\lim_{h \to 0} \phi(h)$ *does not exist. Therefore, G is not differentiable at* $x = 0$*.*

The function F given in (2) is also defined in every interval that contains 0*. Hence, F is differentiable at* 0 *if and only if*

$$\lim_{h \to 0} \frac{F(0+h) - F(0)}{h} = \lim_{h \to 0} \frac{h^3 \sin \frac{1}{h} - 0}{h} = \lim_{h \to 0} \left(h^2 \sin \frac{1}{h} \right)$$

exists. Now we know that, for $h \neq 0$*,* $\left| \sin \frac{1}{h} \right| \leq 1$ *and* $\lim_{h \to 0} h^2 = 0$*; it follows from a simple modification of what was proved in Exercise #6 of Problem Set D that* $\lim_{h \to 0} \left(h^2 \sin \frac{1}{h} \right) = 0$*. Therefore, F is differentiable at* $x = 0$ *and* $F'(0) = 0$*.*

Excursion 6.1.10 *In the space provided, sketch graphs of G and F on two different representations of the Cartesian coordinate system in intervals containing* 0.

For the sketch of G using the curves $y = x$ and $y = -x$ as guides to stay within should have helped give a nice sense for the appearance of the graph; the guiding (or bounding) curves for F are $y = x^3$ and $y = -x^3$.

Remark 6.1.11 *The two problems done in the last example illustrate what is sometimes referred to as a smoothing effect. In our text, it is shown that*

$$K(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{, for } \quad x \neq 0 \\ \\ 0 & \text{, for } \quad x = 0 \end{cases}$$

is also differentiable at $x = 0$. The function

$$L(x) = \begin{cases} \sin \dfrac{1}{x} & \text{, for } \quad x \neq 0 \\ \\ 0 & \text{, for } \quad x = 0 \end{cases}$$

is not continuous at $x = 0$ with the discontinuity being of the second kind. The "niceness" of the function is improving with the increase in exponent of the "smoothing function" x^n.

In the space provided, sketch graphs of K *and* L *on two different representations of the Cartesian coordinate system in intervals containing* 0*.*

The function L *is not continuous at* $x = 0$ *while* G *is continuous at* $x = 0$ *but not differentiable there. Now we know that* K *and* F *are both differentiable at* $x = 0$*; in fact, it can be shown that* F *can be defined to be differentiable at* $x = 0$ *while at most continuity at* $x = 0$ *can be gained for the derivative of* K *at* $x = 0$*. Our first theorem in this section will justify the claim that being differentiable is a stronger condition than being continuous; this offers one sense in which we claim that* F *is a nicer function in intervals containing* 0 *than* K *is there.*

Excursion 6.1.12 *Fill in what is missing in order to complete the following proof that the function* $f(x) = \sqrt{x}$ *is differentiable in* $\mathbb{R}^+ = (0, \infty)$*.*

Proof. *Let* $f(x) = \sqrt{x}$ *and suppose that* $a \in \mathbb{R}^+$*. Then* f *is* $\underset{(1)}{\underline{\qquad\qquad\qquad}}$ *in the segment* $\left(\frac{a}{2}, 2a\right)$ *that contains* $x = a$*. Hence,* f *is differentiable at* $x = a$ *if and only if*

$$\lim_{h \to 0} \underset{(2)}{\boxed{\qquad\qquad\qquad\qquad}} = \lim_{h \to 0} \underset{(3)}{\boxed{\qquad\qquad\qquad\qquad}}$$

exists. Now

$$\begin{aligned}\lim_{h\to 0}\underset{(3)}{\boxed{\hspace{5cm}}} &= \lim_{h\to 0}\left[\frac{\left(\sqrt{a+h}-\sqrt{a}\right)\left(\sqrt{a+h}+\sqrt{a}\right)}{h\left(\sqrt{a+h}+\sqrt{a}\right)}\right]\\ &= \underset{(4)}{\underline{\hspace{7cm}}}\\ &= \underset{(5)}{\underline{\hspace{7cm}}}\\ &= \underset{(6)}{\underline{\hspace{7cm}}}.\end{aligned}$$

Consequently, f is differentiable at $x=a$ and $f'(a)=\underset{(7)}{\underline{\hspace{1.5cm}}}$. Since $a\in\mathbb{R}^+$ was arbitrary, we conclude that

$$(\forall x)\left[\left(x\in\mathbb{R}^+\wedge f(x)=\sqrt{x}\right)\Rightarrow f'(x)=\frac{1}{2\sqrt{x}}\right].$$

■

Acceptable responses are: (1) defined, (2) $\left[\left(f(a+h)-f(a)\right)\left(h^{-1}\right)\right]$, (3) $\left[\left(\sqrt{a+h}-\sqrt{a}\right)\left(h^{-1}\right)\right]$, (4) $\lim_{h\to 0}\left[\frac{(a+h)-a}{h\left(\sqrt{a+h}+\sqrt{a}\right)}\right]$, (5) $\lim_{h\to 0}\left(\sqrt{a+h}+\sqrt{a}\right)^{-1}$, (6) $\left(2\sqrt{a}\right)^{-1}$, and (7) $\frac{1}{2\sqrt{a}}$.\

The next result tells us that differentiability of a function at a point is a stronger condition than continuity at the point.

Theorem 6.1.13 *If a function is differentiable at $\zeta\in\mathbb{R}$, then it is continuous there.*

Excursion 6.1.14 *Make use of the following observations and your understanding of properties of limits of functions to prove Theorem 6.1.13*

Some observations to ponder:

- *The function f being differentiable at ζ assures the existence of a $\delta>0$ such that f is defined in the segment $(\zeta-\delta,\zeta+\delta)$;*

- *Given a function G defined in a segment (a, b), we know that G is continuous at any point $p \in (a, b)$ if and only if $\lim_{x \to p} G(x) = G(p)$ which is equivalent to having $\lim_{x \to p} \left[G(x) - G(p)\right] = 0$.*

Space for scratch work.

Proof.

$\blacksquare$

Once you think of the possibility of writing $\left[G(x) - G(p)\right]$ as $\left[(G(x) - G(p))(x - p)^{-1}\right](x - p)$ for $x \neq p$ the limit of the product theorem does the rest of the work.

Remark 6.1.15 *We have already seen two examples of functions that are continuous at a point without being differentiable at the point; namely, $g(x) = |x - 2|$ at $x = 2$ and, for $x = 0$,*

$$G(x) = \begin{cases} x \sin \dfrac{1}{x} & , \textit{for} \quad x \neq 0 \\ \\ 0 & , \textit{for} \quad x = 0 \end{cases} .$$

To see that G is continuous at $x = 0$, *note that* $\left|\sin\frac{1}{x}\right| \leq 1$ *for* $x \neq 0$ *and* $\lim_{x\to 0} x = 0$ *implies that* $\lim_{x\to 0}\left((x)\left(\sin\frac{1}{x}\right)\right) = 0$. *Alternatively, for* $\varepsilon > 0$, *let* $\delta(\varepsilon) = \varepsilon$; *then* $0 < |x - 0| < \delta$ *implies that*

$$\left|x \sin\frac{1}{x} - 0\right| = |x|\left|\sin\frac{1}{x}\right| \leq |x| < \delta = \varepsilon.$$

Hence, $\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0 = G(0)$. *Either example is sufficient to justify that the converse of Theorem 6.1.13 is not true.*

Because the derivative is defined as the limit of the difference quotient, it should come as no surprise that we have a set of properties involving the derivatives of functions that follow directly and simply from the definition and application of our limit theorems. The set of basic properties is all that is needed in order to make a transition from finding derivatives using the definition to finding derivatives using simple algebraic manipulations.

Theorem 6.1.16 (Properties of Derivatives) *(a) If c is a constant function, then* $c'(x) = 0$.

(b) If f *is differentiable at* ζ *and* k *is a constant, then* $h(x) = kf(x)$ *is differentiable at* ζ *and* $h'(\zeta) = kf'(\zeta)$.

(c) If f *and* g *are differentiable at* ζ, *then* $F(x) = (f+g)(x)$ *is differentiable at* ζ *and* $F'(\zeta) = f'(\zeta) + g'(\zeta)$.

(d) If u *and* v *are differentiable at* ζ, *then* $G(x) = (uv)(x)$ *is differentiable at* ζ *and*

$$G'(\zeta) = u(\zeta)v'(\zeta) + v(\zeta)u'(\zeta).$$

(e) If f *is differentiable at* ζ *and* $f(\zeta) \neq 0$, *then* $H(x) = [f(x)]^{-1}$ *is differentiable at* ζ *and* $H'(\zeta) = -\frac{f'(\zeta)}{[f(\zeta)]^2}$.

(f) If $p(x) = x^n$ *for* n *an integer,* p *is differentiable wherever it is defined and*

$$p'(x) = nx^{n-1}.$$

The proofs of (a) and (b) are about as easy as it gets while the straightforward proofs of (c) and (f) are left as exercises. Completing the next two excursions will provide proofs for (d) and (e).

Excursion 6.1.17 *Fill is what is missing in order to complete the following proof that, if u and υ are differentiable at ζ, then $G(x) = (u\upsilon)(x)$ is differentiable at ζ and*

$$G'(\zeta) = u(\zeta)\upsilon'(\zeta) + \upsilon(\zeta)u'(\zeta).$$

Proof. *Suppose u, υ, and G are as described in the hypothesis. Because u and υ are differentiable at ζ, they are defined in a segment containing ζ. Hence, $G(x) = u(x)\,\upsilon(x)$ is defined in a segment containing ζ. Hence, G is differentiable at ζ if and only if $\lim\limits_{h\to 0}\underset{(1)}{\boxed{}}$ exists. Note that*

$$\begin{aligned}
\lim_{h\to 0}\underset{(1)}{\boxed{}} &= \lim_{h\to 0}\underset{(2)}{\boxed{}} \\
&= \lim_{h\to 0}\frac{\upsilon(\zeta+h)\left[u(\zeta+h)-u(\zeta)\right]+u(\zeta)\left[\upsilon(\zeta+h)-\upsilon(\zeta)\right]}{h} \\
&= \lim_{h\to 0}\left[\upsilon(\zeta+h)\left(\frac{u(\zeta+h)-u(\zeta)}{h}\right)+u(\zeta)\left(\frac{\upsilon(\zeta+h)-\upsilon(\zeta)}{h}\right)\right].
\end{aligned}$$

Since υ is differentiable at ζ it is continuous there; thus, $\lim\limits_{h\to 0}\upsilon(\zeta+h) = \underset{(3)}{\underline{\qquad\qquad}}$.

Now the differentiability of u and υ with the limit of the product and limit of the sum theorems yield that

$$\lim_{h\to 0}\underset{(1)}{\boxed{}} = \underset{(4)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}.$$

Therefore, G is differentiable at ζ. ■

Acceptable responses are: (1) $\left[(G(\zeta+h)-G(\zeta))\,h^{-1}\right]$, (2) $\left[(u(\zeta+h)\,\upsilon(\zeta+h)-u(\zeta)\,\upsilon(\zeta))\,h^{-1}\right]$, (3) $\upsilon(\zeta)$, and (4) $\upsilon(\zeta)\,u'(\zeta)+u(\zeta)\,\upsilon'(\zeta)$.

Excursion 6.1.18 *Fill is what is missing in order to complete the following proof that, if* f *is differentiable at* ζ *and* $f(\zeta) \neq 0$*, then* $H(x) = [f(x)]^{-1}$ *is differentiable at* ζ *and* $H'(\zeta) = -\dfrac{f'(\zeta)}{[f(\zeta)]^2}$.

Proof. *Suppose that the function* f *is differentiable at* ζ *and* $f(\zeta) \neq 0$*. From Theorem 6.1.13,* f *is* $\underset{(1)}{\underline{\hspace{4cm}}}$ *at* ζ*. Hence.* $\lim_{x\to\zeta} f(x) = \underset{(2)}{\underline{\hspace{2cm}}}$*. Since* $\varepsilon = \dfrac{|f(\zeta)|}{2} > 0$*, it follows that there exists* $\delta > 0$ *such that* $\underset{(3)}{\underline{\hspace{8cm}}}$ *implies that* $|f(x) - f(\zeta)| < \dfrac{|f(\zeta)|}{2}$*. The (other) triangular inequality, yields that, for* $\underset{(3)}{\underline{\hspace{8cm}}}$, $|f(\zeta)| - |f(x)| < \dfrac{|f(\zeta)|}{2}$ *from which we conclude that* $|f(x)| > \dfrac{|f(\zeta)|}{2}$ *in the segment* $\underset{(4)}{\underline{\hspace{4cm}}}$*. Therefore, the function* $H(x) \underset{def}{=} [f(x)]^{-1}$ *is defined in a segment that contains* ζ *and it is differentiable at* ζ *if and only if* $\lim_{h\to 0} \dfrac{H(\zeta+h) - H(\zeta)}{h}$ *exists. Now simple algebraic manipulations yield that*

$$\lim_{h\to 0} \frac{H(\zeta+h) - H(\zeta)}{h} = \lim_{h\to 0} \left[\left(\frac{f(\zeta+h) - f(\zeta)}{h} \right) \left(\frac{-1}{f(\zeta+h)\, f(\zeta)} \right) \right].$$

From the $\underset{(5)}{\underline{\hspace{4cm}}}$ *of* f *at* ζ*, it follows that* $\lim_{h\to 0} f(\zeta+h) = \underset{(6)}{\underline{\hspace{4cm}}}$*. In view of the differentiability of* f *and the limit of the product theorem, we have that*

$$\lim_{h\to 0} \frac{H(\zeta+h) - H(\zeta)}{h} = \underset{(7)}{\underline{\hspace{4cm}}}.$$

■

Acceptable responses are: (1) continuous, (2) $f(\zeta)$, (3) $|x - \zeta| < \delta$, (4) $(\zeta - \delta, \zeta + \delta)$, (5) continuity, (6) $f(\zeta)$, and (7) $-\left(f'(\zeta)\right)[f(\zeta)]^{-2}$.

The next result offers a different way to think of the difference quotient.

Theorem 6.1.19 (Fundamental Lemma of Differentiation) *Suppose that f is differentiable at x_0. Then there exists a function η defined on an open interval containing 0 for which $\eta(0) = 0$ and*

$$f(x_0 + h) - f(x_0) = [f'(x_0) + \eta(h)] \cdot h \tag{6.3}$$

and η is continuous at 0.

Before looking at the proof take a few moments to reflect on what you can say about

$$\frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0)$$

for $|h| > 0$.

Proof. Suppose that $\delta > 0$ is such that f is defined in $|x - x_0| < \delta$ and let

$$\eta(h) = \begin{cases} \dfrac{1}{h}\left[f(x_0 + h) - f(x_0)\right] - f'(x_0) & \text{, if } 0 < |h| < \delta \\ 0 & \text{, if } h = 0 \end{cases} .$$

Because f is differentiable at x_0, it follows from the limit of the sum theorem that $\lim_{h \to 0} \eta(h) = 0$. Since $\eta(0) = 0$, we conclude that η is continuous at 0. Finally, solving $\eta = \dfrac{1}{h}\left[f(x_0 + h) - f(x_0)\right] - f'(x_0)$ for $f(x_0 + h) - f(x_0)$ yields (6.3). ■

Remark 6.1.20 *If f is differentiable at x_0, then*

$$f(x_0 + h) \approx f(x_0) + f'(x_0)h$$

for h very small; i.e., the function near to x_0 is approximated by a linear function whose slope is $f'(x_0)$.

Next, we will use the Fundamental Lemma of Differentiation to obtain the derivative of the composition of differentiable functions.

Theorem 6.1.21 (Chain Rule) *Suppose that g and u are functions on $\mathbb{R}$ and that $f(x) = g(u(x))$. If u is differentiable at x_0 and g is differentiable at $u(x_0)$, then f is differentiable at x_0 and*

$$f'(x_0) = g'(u(x_0)) \cdot u'(x_0).$$

****

Before reviewing the offered proof, look at the following and think about what prompted the indicated rearrangement; What should be put in the boxes to enable us to relate to the given information?

We want to consider

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \\
&\quad = \lim_{h\to 0} \frac{g\left(u(x_0+h)\right) - g\left(u(x_0)\right)}{h} \\
&\quad = \lim_{h\to 0} \left(\frac{g\left(u(x_0+h)\right) - g\left(u(x_0)\right)}{\boxed{\qquad\qquad\qquad}} \cdot \frac{\boxed{\qquad\qquad\qquad}}{h} \right)
\end{aligned}$$

****

Proof. Let $\Delta f = f(x_0+h) - f(x_0)$, $\Delta u = u(x_0+h) - u(x_0)$ and $u_0 = u(x_0)$. Then

$$\Delta f = g(u(x_0+h)) - g(u(x_0)) = g(u_0 + \Delta u) - g(u_0).$$

Because u is continuous at x_0, we know that $\lim_{h\to 0} \Delta u = 0$. By the Fundamental Lemma of Differentiation, there exists a function η, with $\eta(0) = 0$, that is continuous at 0 and is such that $\Delta f = [g'(u_0) + \eta(\Delta u)]\Delta u$. Hence,

$$\lim_{h\to 0} \frac{\Delta f}{h} = \lim_{h\to 0} \left([g'(u_0) + \eta(\Delta u)] \frac{\Delta u}{h} \right) = g'(u_0)\, u'(x_0)$$

from the limit of the sum and limit of the product theorems. ■

6.1.1 Formulas for Differentiation

As a consequence of the results in this section, we can justify the differentiation of all polynomials and rational functions. From Excursion 6.1.12, we know that the formula given in the Properties of Derivatives Theorem (f) is valid for $n = \frac{1}{2}$. In fact, it is valid for all nonzero real numbers. Prior to the Chain Rule, the only way to find the derivative of $f(x) = \left(x^3 + \left(3x^2 - 7\right)^{12}\right)^8$, other than appeal to the definition, was to expand the expression and apply the Properties of Derivatives Theorem, parts (a), (b), (c) and (f); in view of the Chain Rule and the Properties of Derivatives Theorem, we have

$$f'(x) = 8\left(x^3 + \left(3x^2 - 7\right)^{12}\right)^7 \left[3x^2 + 72x\left(3x^2 - 7\right)^{11}\right].$$

What we don't have yet is the derivatives of functions that are not realized as algebraic combinations of polynomials; most notably this includes the trigonometric functions, the inverse trig functions, α^x for any fixed positive real number α, and the logarithm functions.

For any $x \in \mathbb{R}$, we know that

$$\begin{aligned}\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} &= \lim_{h\to 0} \frac{\sin(h)\cos(x) + \cos(h)\sin x - \sin x}{h} \\ &= \lim_{h\to 0}\left[(\cos x)\left(\frac{\sin(h)}{h}\right) + (\sin x)\left(\frac{\cos(h) - 1}{h}\right)\right]\end{aligned}$$

and

$$\begin{aligned}\lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} &= \lim_{h\to 0} \frac{\cos(h)\cos(x) - \sin(h)\sin x - \cos x}{h} \\ &= \lim_{h\to 0}\left[(\cos x)\left(\frac{\cos(h) - 1}{h}\right) - (\sin x)\left(\frac{\sin(h)}{h}\right)\right].\end{aligned}$$

Consequently, in view of the limit of the sum and limit of the product theorems, finding the derivatives of the sine and cosine functions depends on the existence of $\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)$ and $\lim_{h\to 0}\left(\frac{\cos(h) - 1}{h}\right)$. Using elementary geometry and trigonometry, it can be shown that the values of these limits are 1 and 0, respectively. An outline for the proofs of these two limits, which is a review of what is shown in an elementary calculus course, is given as an exercise. The formulas for the derivatives of the other trigonometric functions follow as simple applications of the Properties of Derivatives.

Recall that $e = \lim_{\zeta \to 0} (1+\zeta)^{1/\zeta}$ and $y = \ln x \Leftrightarrow x = e^y$. With these in addition to basic properties of logarithms, for x a positive real,

$$\begin{aligned}\lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} &= \lim_{h\to 0} \left[\frac{1}{h} \ln\left(1+\frac{h}{x}\right)\right] \\ &= \lim_{h\to 0} \left[\ln\left(1+\frac{h}{x}\right)^{1/h}\right].\end{aligned}$$

Keeping in mind that x is a constant, it follows that

$$\begin{aligned}\lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} &= \lim_{h\to 0} \left[\ln\left[\left(1+\frac{h}{x}\right)^{x/h}\right]^{1/x}\right] \\ &= \frac{1}{x}\lim_{h\to 0} \left[\ln\left[\left(1+\frac{h}{x}\right)^{x/h}\right]\right]\end{aligned}$$

Because $\left[\left(1+\frac{h}{x}\right)^{x/h}\right] \longrightarrow e$ as $h \longrightarrow 0$ and $\ln(e) = 1$, the same argument that was used for the proof of Theorem 5.2.11 allows us to conclude that

$$\lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} = \frac{1}{x}.$$

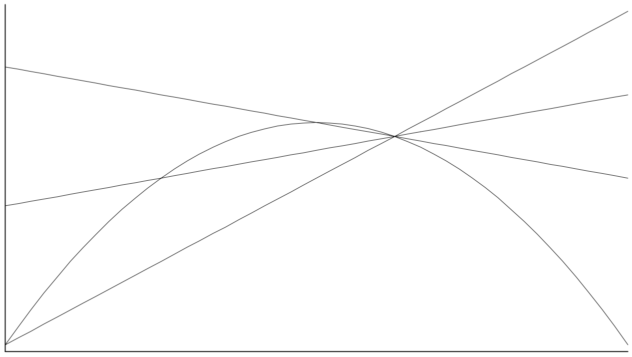
Formulas for the derivatives of the inverse trigonometric functions and α^x, for any fixed positive real number α, will follow from the theorem on the derivative of the inverses of a function that is proved at the end of this chapter.

6.1.2 Revisiting A Geometric Interpretation for the Derivative

Completing the following figure should serve as a nice reminder of one of the common interpretations and applications of the derivative of a function at the point.

- On the x-axis, label the x-coordinate of the common point of intersection of the curve, $f(x)$, and the three indicated lines as c.

- Corresponding to each line–ℓ_1, ℓ_2, and ℓ_3, on the x-axis label the x-coordinate of the common point of intersection of the curve, $f(x)$, with the line as $c+h_1$, $c+h_2$, and $c+h_3$ in ascending order. Note that h_1, h_2 and h_3 are negative in the set-up that is shown. Each of the lines ℓ_1, ℓ_2, and ℓ_3 are called **secant lines**.
- Find the slopes m_1, m_2, and m_3, respectively, of the three lines.



Excursion 6.1.22 *Using terminology associated with the derivative, give a brief description that applies to each of the slopes* m_j *for* $j = 1, 2, 3$.

Excursion 6.1.23 *Give a concise well-written description of the geometric interpretation for the derivative of* f *at* $x = c$*, if it exists.*

6.2 The Derivative and Function Behavior

The difference quotient is the ratio of the change in function values to the change in arguments. Consequently, it should come as no surprise that the derivative provides information related to monotonicity of functions.

In the following, continuity on an interval $I = [a, b]$ is equivalent to having continuity on (a, b), right-hand continuity at $x = a$ and left-hand continuity at $x = b$. For right-hand continuity at $x = a$; $f(a+) = f(a)$, while left-hand continuity at $x = b$ requires that $f(b-) = f(b)$.

Definition 6.2.1 *A real valued function f on a metric space (X, d_X) has a* ***local maximum*** *at a point $p \in X$ if and only if*

$$(\exists \delta > 0)\left[(\forall q)\left(q \in N_\delta(p) \Rightarrow f(q) \le f(p)\right)\right];$$

the function has a ***local minimum*** *at a point $p \in X$ if and only if*

$$(\exists \delta > 0)\left[(\forall q)\left(q \in N_\delta(p) \Rightarrow f(p) \le f(q)\right)\right].$$

Definition 6.2.2 *A real valued function f on a metric space (X, d_X) has a* ***(global) maximum*** *at a point $p \in X$ if and only if*

$$\left[(\forall x)\left(x \in X \Rightarrow f(x) \le f(p)\right)\right];$$

the function has a ***(global) minimum*** *at a point $p \in X$ if and only if*

$$\left[(\forall x)\left(q \in X \Rightarrow f(p) \le f(x)\right)\right].$$

Theorem 6.2.3 (Interior Extrema Theorem) *Suppose that f is a function that is defined on an interval $I = [a, b]$. If f has a local maximum or local minimum at a point $x_0 \in (a, b)$ and f is differentiable at x_0, then $f'(x_0) = 0$.*

Space for scratch work or motivational picture.

Proof. Suppose that the function f is defined in the interval $I = [a, b]$, has a local maximum at $x_0 \in (a, b)$, and is differentiable at x_0. Because f has a local maximum at x_0, there exists a positive real number δ such that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ and $(\forall t)\left[t \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(t) \le f(x_0)\right]$. Thus, for $t \in (x_0 - \delta, x_0)$,

$$\frac{f(t) - f(x_0)}{t - x_0} \ge 0 \tag{6.4}$$

while $t \in (x_0, x_0 + \delta)$ implies that

$$\frac{f(t) - f(x_0)}{t - x_0} \leq 0. \tag{6.5}$$

Because f is differential at x_0, $\lim\limits_{t \to x_0} \dfrac{f(t) - f(x_0)}{t - x_0}$ exists and is equal to $f'(x_0)$. From (6.4) and (6.5), we know that $f'(x_0) \geq 0$ and $f'(x_0) \leq 0$, respectively. The Trichotomy Law yields that $f'(x_0) = 0$. ■

The Generalized Mean-Value Theorem that follows the next two results contains Rolle's Theorem and the Mean-Value Theorem as special cases. We offer the results in this order because it is easier to appreciate the generalized result after reflecting upon the geometric perspective that is offered by the two lemmas.

Lemma 6.2.4 (Rolle's Theorem) *Suppose that f is a function that is continuous on the interval $I = [a, b]$ and differentiable on the segment $I^\circ = (a, b)$. If $f(a) = f(b)$, then there is a number $x_0 \in I^\circ$ such that $f'(x_0) = 0$.*

Space for scratch work or building intuition via a typical picture.

Proof. If f is constant, we are done. Thus, we assume that f is not constant in the interval (a, b). Since f is continuous on I, by the Extreme Value Theorem, there exists points ζ_0 and ζ_1 in I such that

$$f(\zeta_0) \leq f(x) \leq f(\zeta_1) \text{ for all } x \in I.$$

Because f is not constant, at least one of $\{x \in I : f(x) > f(a)\}$ and

$\{x \in I : f(x) < f(a)\}$ is nonempty. If $\{x \in I : f(x) > f(a)\} = (a, b)$, then $f(\zeta_0) = f(a) = f(b)$ and, by the Interior Extrema Theorem, $\zeta_1 \in (a, b)$ is such that $f'(\zeta_1) = 0$. If $\{x \in I : f(x) < f(a)\} = (a, b)$, then $f(\zeta_1) = f(a) = f(b)$, $\zeta_0 \in (a, b)$, and the Interior Extrema Theorem implies that $f'(\zeta_0) = 0$. Finally, if $\{x \in I : f(x) > f(a)\} \neq (a, b)$ and $\{x \in I : f(x) < f(a)\} \neq (a, b)$, then both ζ_0 and ζ_1 are in (a, b) and $f'(\zeta_0) = f'(\zeta_1) = 0$. ■

Lemma 6.2.5 (Mean-Value Theorem) *Suppose that f is a function that is continuous on the interval $I = [a, b]$ and differentiable on the segment $I^\circ = (a, b)$. Then* *there exists a number $\xi \in I^\circ$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Excursion 6.2.6 *Use the space provided to complete the proof of the Mean-Value Theorem.*

Proof. *Consider the function F defined by*

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

as a candidate for application of Rolle's Theorem.

■

Theorem 6.2.7 (Generalized Mean-Value Theorem) *Suppose that f and F are functions that are continuous on the interval $I = [a, b]$ and differentiable on the segment I°. If $F'(x) \neq 0$ on I°, then*

(a) $F(b) - F(a) \neq 0$, *and*

(b) $(\exists \xi)\left(\xi \in I^\circ \wedge \dfrac{f(b) - f(a)}{F(b) - F(a)} = \dfrac{f'(\xi)}{F'(\xi)}\right)$.

Excursion 6.2.8 *Fill in the indicated steps in order to complete the proof of the Generalized Mean-Value Theorem.*

Proof. To complete a proof of (a), apply the Mean-Value Theorem to F.

For (b), for $x \in I$, define the function by

$$\varphi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{F(b) - F(a)} \cdot [F(x) - F(a)].$$

It follows directly that $\varphi(a) = \varphi(b) = 0$.

■

Theorem 6.2.9 (Monotonicity Test) *Suppose that a function f is differentiable in the segment (a, b).*

(a) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing in (a, b).

(b) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant in (a, b).

(c) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing in (a, b).

Excursion 6.2.10 *Fill in what is missing in order to complete the following proof of the Monotonicity Test.*

Proof. *Suppose that f is differentiable in the segment (a, b) and $x_1, x_2 \in (a, b)$ are such that $x_1 < x_2$. Then f is continuous in $[x_1, x_2]$ and* $\underset{(1)}{\underline{\hspace{6cm}}}$

in (x_1, x_2). *From the* $\underset{(2)}{\underline{\hspace{6cm}}}$, *there exists* $\xi \in (x_1, x_2)$ *such that*

$$f'(\xi) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

If $f'(x) \geq 0$ *for all* $x \in (a, b)$, *then* $f'(\xi) \geq 0$. *Since* $x_1 - x_2 < 0$, *it follows that* $\underset{(3)}{\underline{\hspace{6cm}}}$; *i.e.,* $f(x_1) \leq f(x_2)$. *Since* x_1 *and* x_2 *were arbitrary, we have that*

$$(\forall x_1)(\forall x_2)\left[\left(x_1, x_2 \in (a, b) \wedge \underset{(4)}{\underline{\hspace{3cm}}}\right) \Rightarrow f(x_1) \leq f(x_2)\right].$$

Hence, f *is* $\underset{(5)}{\underline{\hspace{6cm}}}$ *in* (a, b).

If $f'(x) = 0$ *for all* $x \in (a, b)$, *then*

$$\underset{(6)}{\underline{\hspace{9cm}}}.$$

Finally, if $f'(x) \leq 0$ *for all* $x \in (a, b)$,

$$\underset{(7)}{\underline{\hspace{9cm}}} \quad \blacksquare$$

Acceptable responses are: (1) differentiable, (2) Mean-Value Theorem, (3) $f(x_1) - f(x_2) \leq 0$, (4) $x_1 < x_2$, (5) monotonically increasing, (6) $f(x_1) - f(x_2) = 0$; i.e., $f(x_1) = f(x_2)$. Since x_1 and x_2 were arbitrary, we have that f is constant throughout (a, b)., (7) then $f'(\xi) \leq 0$ and $x_1 - x_2 < 0$ implies that $f(x_1) - f(x_2) \geq 0$; i.e., $f(x_1) \geq f(x_2)$. Because x_1 and x_2 were arbitrary we conclude that f is monotonically decreasing in (a, b).

Example 6.2.11 *Discuss the monotonicity of* $f(x) = 2x^3 + 3x^2 - 36x + 7$.

For $x \in \mathbb{R}$, $f'(x) = 6x^2 + 6x - 36 = 6(x + 3)(x - 2)$. *Since* f' *is positive in* $(-\infty, -3)$ *and* $(2, \infty)$, f *is monotonically increasing there, while* f' *negative in* $(-3, 2)$ *yields that* f *is monotonically decreasing in that segment.*

Remark 6.2.12 *Actually, in each of open intervals* $(-\infty,-3)$, $(2,\infty)$, *and* $(-3,2)$ *that were found in Example 6.2.11, we have strict monotonicity; i.e., for* $x_1, x_2 \in (-\infty,-3)$ *or* $x_1, x_2 \in (2,\infty)$, $x_1 < x_2$ *implies that* $f(x_1) < f(x_2)$, *while* $x_1, x_2 \in (-3,2)$ *and* $x_1 < x_2$ *yields that* $f(x_1) > f(x_2)$.

6.2.1 Continuity (or Discontinuity) of Derivatives

Given a real-valued function f that is differentiable on a subset $\boldsymbol{\Omega}$ of $\mathbb{R}$, the derivative $F = f'$ is a function with domain $\boldsymbol{\Omega}$. We have already seen that F need not be continuous. It is natural to ask if there are any nice properties that can be associated with the derivative. The next theorem tells us that the derivative of a real function that is differentiable on an interval satisfies the intermediate value property there.

Theorem 6.2.13 *Suppose that* f *is a real valued function that is differentiable on* $[a,b]$ *and* $f'(a) < f'(b)$. *Then for any* $\lambda \in \mathbb{R}$ *such that* $f'(a) < \lambda < f'(b)$, *there exists a point* $x \in (a,b)$ *such the* $f'(x) = \lambda$.

Proof. *Suppose that* f *is a real valued function that is differentiable on* $[a,b]$ *and* $\lambda \in \mathbb{R}$ *is such that* $f'(a) < \lambda < f'(b)$. *Let* $G(t) = f(t) - \lambda t$. *From the Properties of Derivatives, G is differentiable on* $[a,b]$. *By Theorem 6.1.13, G is continuous on* $[a,b]$ *from which the Extreme Value Theorem yields that G has a minimum at some* $x \in [a,b]$. *Since* $G'(a) = f'(t) - \lambda < 0$ *and* $G'(b) = f'(t) - \lambda > 0$, *there exists a* $t_1 \in (a,b)$ *and* $t_2 \in (a,b)$ *such that* $G(t_1) < G(a)$ *and* $G(t_2) < G(b)$. *It follows that neither* $(a, G(a))$ *nor* $(b, G(b))$ *is a minimum of G in* $[a,b]$. *Thus,* $a < x < b$. *In view of the Interior Extrema Theorem, we have that* $G'(x) = 0$ *which is equivalent to* $f'(x) = \lambda$ ■

Remark 6.2.14 *With the obvious algebraic modifications, it can be shown that the same result holds if the real valued function that is differentiable on* $[a,b]$ *satisfies* $f'(a) > f'(b)$.

Corollary 6.2.15 *If* f *is a real valued function that is differentiable on* $[a,b]$, *then* f' *cannot have any simple(first kind) discontinuities on* $[a,b]$.

Remark 6.2.16 *The corollary tells us that any discontinuities of real valued functions that are differentiable on an interval will have only discontinuities of the second kind.*

6.3 The Derivative and Finding Limits

The next result allows us to make use of derivatives to obtain some limits: It can be used to find limits in the situations for which we have been using the Limit of Almost Equal Functions and to find some limits that we have not had an easy means of finding.

Theorem 6.3.1 (L'Hôpital's Rule I) *Suppose that f and F are functions such that f' and F' exist on a segment $I = (a, b)$ and $F' \neq 0$ on I.*

(a) If $f(a+) = F(a+) = 0$ and $\left(\frac{f'}{F'}\right)(a+) = L$, then $\left(\frac{f}{F}\right)(a+) = L$.

(b) If $f(a+) = F(a+) = \infty$ and $\left(\frac{f'}{F'}\right)(a+) = L$, then $\left(\frac{f}{F}\right)(a+) = L$.

Excursion 6.3.2 *Fill in what is missing in order to compete the following proof of part (a).*

Proof. *Suppose that f and F are differentiable on a segment $I = (a, b)$, $F' \neq 0$ on I, and $f(a+) = F(a+) = 0$. Setting $f(a) = f(a+)$ and $F(a) = F(a+)$ extends f and F to functions that are $\underset{(1)}{\underline{\qquad\qquad\qquad}}$ in $[a, b)$. With this, $F(a) = 0$ and $F'(x) \neq 0$ in I yields that $F(x)$ $\underset{(2)}{\underline{\qquad\quad}}$.*

Suppose that $\varepsilon > 0$ is given. Since $\left(\frac{f'}{F'}\right)(a+) = L$, there exists $\delta > 0$ such that $a < w < a + \delta$ implies that

$$\underset{(2)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}$$

From the Generalized Mean-Value Theorem and the fact that $F(a) = f(a) = 0$, it follows that

$$\left|\frac{f(x)}{\underbrace{\qquad}_{(3)}} - L\right| = \left|\frac{\overbrace{\qquad}^{(4)}}{F(x) - F(a)} - L\right| = \left|\frac{\qquad}{\underbrace{\qquad\quad}_{(5)}} - L\right|$$

for some ξ *satisfying* $a < \xi < a + \delta$. *Hence,* $\left|\dfrac{f(x)}{F(x)} - L\right| < \varepsilon$. *Since* $\varepsilon > 0$ *was arbitrary, we conclude that* $\underset{(6)}{\underline{\hspace{6cm}}}$. ■

Acceptable responses are: (1) continuous, (2) $\neq 0$, (3) $\left|\dfrac{f'(w)}{F'(w)} - L\right| < \varepsilon$, (4) $F(x)$, (5) $f(x) - f(a)$, (6) $\dfrac{f'(\xi)}{F'(\xi)}$, and (7) $\lim\limits_{x \to a^+} \dfrac{f(x)}{F(x)} = L$.

Proof. Proof of (b). Suppose that f and F are functions such that f' and F' exist on an open interval $I = \{x : a < x < b\}$, $F' \neq 0$ on I, $f(a+) = F(a+) = \infty$ and $\left(\dfrac{f'}{F'}\right)(a+) = L$. Then f and F are continuous on I and there exists $h > 0$ such that $F' \neq 0$ in $I_h = \{x : a < x < a + h\}$. For $\epsilon > 0$, there exists a δ with $0 < \delta < h$ such that

$$\left|\frac{f'(\xi)}{F'(\xi)} - L\right| < \frac{\epsilon}{2} \text{ for all } \xi \text{ in } I_\delta = \{x : a < x < a + \delta\}.$$

Let x and c be such that $x < c$ and $x, c \in I_\delta$. By the Generalized Mean-Value Theorem, there exists a ξ in I_δ such that $\dfrac{f(x) - f(c)}{F(x) - F(c)} = \dfrac{f'(\xi)}{F'(\xi)}$. Hence,

$$\left|\frac{f(x) - f(c)}{F(x) - F(c)} - L\right| < \frac{\epsilon}{2}.$$

In particular, for $\epsilon < 1$, we have that

$$\left|\frac{f(x) - f(c)}{F(x) - F(c)}\right| = \left|\frac{f(x) - f(c)}{F(x) - F(c)} - L + L\right| < |L| + \frac{1}{2}.$$

With a certain amount of playing around we claim that

$$\begin{aligned}\left|\frac{f(x)}{F(x)} - \frac{f(x) - f(c)}{F(x) - F(c)}\right| &= \left|\frac{f(c)}{F(x)} - \frac{F(c)}{f(x)} \cdot \frac{f(x) - f(c)}{F(x) - F(c)}\right| \\ &\leq \left|\frac{f(c)}{F(x)}\right| - \left|\frac{F(c)}{f(x)}\right| \left(|L| + \frac{1}{2}\right).\end{aligned}$$

For c fixed, $\frac{f(c)}{F(x)} \to 0$ and $\frac{F(c)}{f(x)} \to 0$ as $x \to a^+$. Hence, there exists δ_1, $0 < \delta_1 < \delta$, such that

$$\left|\frac{f(c)}{F(x)}\right| < \frac{\epsilon}{4} \text{ and } \left|\frac{F(c)}{f(x)}\right| < \frac{1}{4(|L| + 1/2)}\ .$$

Combining the inequalities leads to

$$\left|\frac{f(x)}{F(x)}-L\right| \leq \left|\frac{f(x)}{F(x)}-\frac{f(x)-f(c)}{F(x)-F(c)}\right|+\left|\frac{f(x)-f(c)}{F(x)-F(c)}-L\right|<\epsilon$$

whenever $a < x < a+\delta_1$. Since $\epsilon > 0$ was arbitrary, we conclude that

$$\left(\frac{f}{F}\right)(a+)=\lim_{x\to a^+}\frac{f(x)}{F(x)}=L.$$

■

Remark 6.3.3 *The two statements given in L'Hôpital's Rule are illustrative of the set of such results. For example, the* $x \to a^+$ *can be replaced with* $x \to b^-$, $x \to +\infty$, $x \to \infty$, *and* $x \to -\infty$, *with some appropriate modifications in the statements. The following statement is the one that is given as Theorem 5.13 in our text.*

Theorem 6.3.4 (L'Hôpital's Rule II) *Suppose* f *and* g *are real and differentiable in* (a,b), *where* $-\infty \leq a < b \leq \infty$, $g'(x) \neq 0$ *for all* $x \in (a,b)$, *and* $\lim_{x\to a}\frac{f'(x)}{g'(x)} = A$. *If* $\lim_{x\to a} f(x) = 0 \wedge \lim_{x\to a} g(x) = 0$ *or* $\lim_{x\to a} g(x) = +\infty$, *then* $\lim_{x\to a}\frac{f(x)}{g(x)} = A$.

Excursion 6.3.5 *Use an appropriate form of L'Hôpital's Rule to find*

1. $\lim_{x\to 3}\frac{x^2-5x+6-7\sin(x-3)}{2x-6}$.

2. $\displaystyle\lim_{w\to\infty}\left(1+\frac{1}{w-1}\right)^{w}$

Hopefully, you got -3 and e, respectively.

6.4 Inverse Functions

Recall that for a relation, S, on $\mathbb{R}$, the inverse relation of S, denoted by S^{-1}, is the set of all ordered pairs (y,x) such that $(x,y)\in S$. While a function is a relation that is single-valued, its inverse need not be single-valued. Consequently, we cannot automatically apply the tools of differential calculus to inverses of functions. What follows if some criteria that enables us to talk about "inverse functions." The first result tells us that where a function is increasing, it has an inverse that is a function.

Remark 6.4.1 *If u and v are monotonic functions with the same monotonicity, then their composition (if defined) is increasing. If u and v are monotonic functions with the opposite monotonicity, then their composition (if defined) is decreasing.*

Theorem 6.4.2 (Inverse Function Theorem) *Suppose that f is a continuous function that is strictly monotone on an interval I with $f(I)=J$. Then*

(a) J is an interval;

(b) the inverse relation g of f is a function with domain J that is continuous and strictly monotone on J; and

(c) we have $g(f(x))=x$ for $x\in I$ and $f(g(y))=y$ for $y\in J$.

Proof. Because the continuous image of a connected set is connected and f is strictly monotone, J is an interval. Without loss of generality, we take f to be decreasing in the interval I. Then $f(x_1) \neq f(x_2)$ implies that $x_1 \neq x_2$ and we conclude that, for each w_0 in J, there exists one and only one $\zeta_0 \in I$ such that $w_0 = f(\zeta_0)$. Hence, the inverse of f is a function and the formulas given in (c) hold. It follows from the remark above and (c) that g is strictly decreasing.

To see that g is continuous on J, let w_0 be an interior point of J and suppose that $g(w_0) = x_0$; i.e., $f(x_0) = w_0$. Choose points w_1 and w_2 in J such that $w_1 < w_0 < w_2$. Then there exist points x_1 and x_2 in I, such that $x_1 < x_0 < x_2$, $f(x_1) = w_2$ and $f(x_2) = w_1$. Hence, x_0 is an interior point of I. Now, without loss of generality, take $\epsilon > 0$ small enough that the interval $(x_0 - \epsilon, x_0 + \epsilon)$ is contained in I and define $w_1^* = f(x_0 + \epsilon)$ and $w_2^* = f(x_0 - \epsilon)$ so $w_1^* < w_2^*$. Since g is decreasing,

$$x_0 + \epsilon = g(w_1^*) \geq g(w) \geq g(w_2^*) = x_0 - \epsilon \text{ for } \quad w \text{ such that } \quad w_1^* \leq w \leq w_2^*.$$

Hence,

$$g(w_0) + \epsilon \geq g(w) \geq g(w_0) - \epsilon \text{ for } \quad w \text{ such that } \quad w_1^* \leq w \leq w_2^*.$$

Now taking δ to be the minimum of $w_2^* - w_0$ and $w_0 - w_1^*$ leads to

$$|g(w) - g(w_0)| < \epsilon \text{ whenever } |w - w_0| < \delta.$$

■

Remark 6.4.3 *While we have stated the Inverse Function Theorem in terms of intervals, please note that the term intervals can be replaced by segments (a, b) where a can be $-\infty$ and/or b can be ∞.*

In view of the Inverse Function Theorem, when we have strictly monotone continuous functions, it is natural to think about differentiating their inverses. For a proof of the general result concerning the derivatives of inverse functions, we will make use with the following partial converse of the Chain Rule.

Lemma 6.4.4 *Suppose the real valued functions F, G, and u are such that $F(x) = G(u(x))$, u is continuous at $x_0 \in \mathbb{R}$, $F'(x_0)$ exists, and $G'(u(x_0))$ exists and differs from zero. Then $u'(x_0)$ is defined and $F'(x_0) = G'(u(x_0))\,u'(x_0)$.*

Excursion 6.4.5 *Fill in what is missing to complete the following proof of the Lemma.*

Proof. *Let* $\Delta F = F(x_0+h) - F(x_0)$*,* $\Delta u = u(x_0+h) - u(x_0)$ *and* $u_0 = u(x_0)$*. Then*

$$\Delta F = \underset{(1)}{\underline{\hspace{6cm}}} = G(u_0 + \Delta u) - G(u_0).$$

Since u *is continuous at* x_0*, we know that* $\lim_{h \to 0} \Delta u = 0$*. By the Fundamental Lemma of Differentiation, there exists a function* η*, with* $\underset{(2)}{\underline{\hspace{3cm}}}$*, that is continuous at* 0 *and is such that* $\Delta F = \underset{(3)}{\underline{\hspace{6cm}}}$*. Hence,*

$$\frac{\Delta u}{h} = \frac{\frac{\Delta F}{h}}{[G'(u_0) + \eta(\Delta u)]}.$$

From $\lim_{h \to 0} \Delta u = 0$*, it follows that* $\eta(\Delta u) \longrightarrow 0$ *as* $h \longrightarrow 0$*. Because* $G'(u_0)$ *exists and is nonzero,*

$$u'(x_0) = \lim_{h \to 0} \frac{u(x_0+h) - u(x_0)}{h} = \lim_{h \to 0} \frac{\frac{\Delta F}{h}}{[G'(u_0) + \eta(\Delta u)]} = \frac{F'(x_0)}{G'(u_0)}.$$

Therefore, $u'(x_0)$ *exists and* $\underset{(4)}{\underline{\hspace{6cm}}}$*.* ■

Acceptable responses are: (1) $G(u(x_0+h)) - G(u(x_0))$, (2) $\eta(0) = 0$, (3) $[G'(u_0) + \eta(\Delta u)]\Delta u$, and (4) $F'(x_0) = G'(u_0)\, u'(x_0)$.

Theorem 6.4.6 (Inverse Differentiation Theorem) *Suppose that* f *satisfies the hypotheses of the Inverse Function Theorem. If* x_0 *is a point of* J *such that* $f'(g(x_0))$ *is defined and is different from zero, then* $g'(x_0)$ *exists and*

$$g'(x_0) = \frac{1}{f'(g(x_0))}. \tag{6.6}$$

Proof. From the Inverse Function Theorem, $f(g(x)) = x$. Taking $u = g$ and $G = f$ in Lemma 6.4.4 yields that $g'(x_0)$ exists and $f'(g(x))\, g'(x) = 1$. Since $f'(g(x_0)) \neq 0$, it follows that $g'(x_0) = \frac{1}{f'(g(x_0))}$ as needed. ■

Corollary 6.4.7 *For a fixed nonnegative real number* α*, let* $g(x) = \alpha^x$*. Then* $\operatorname{dom}(g) = \mathbb{R}$ *and, for all* $x \in \mathbb{R}$*,* $g'(x) = \alpha^x \ln \alpha$*.*

Proof. *We know that* $g(x) = \alpha^x$ *is the inverse of* $f(x) = \log_\alpha x$ *where* f *is a strictly increasing function with domain* $(0, \infty)$ *and range* $(-\infty, \infty)$*. Because* $A = \log_\alpha B \Leftrightarrow \alpha^A = B \Leftrightarrow A \ln \alpha = \ln B$*, it follows that*

$$\log_\alpha B = \frac{\ln B}{\ln \alpha}.$$

Hence

$$f'(x) = \left(\log_\alpha x\right)' = \left(\frac{\ln x}{\ln \alpha}\right)' = \frac{1}{x \ln \alpha}.$$

From the Inverse Differentiation Theorem, we have that $g'(x) = \dfrac{1}{f'(g(x))} = g(x) \ln \alpha = \alpha^x \ln \alpha$*.* ■

Remark 6.4.8 *Taking* $\alpha = e$ *in the Corollary yields that* $(e^x)' = e^x$*.*

In practice, finding particular inverses is usually carried out by working directly with the functions given rather than by making a sequence of substitutions.

Example 6.4.9 *Derive a formula, in terms of* x*, for the derivative of* $y = \arctan x$*,* $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$*.*

We know that the inverse of $u = \tan v$ *is a relation that is not a function; consequently we need to restrict ourselves to a subset of the domain. Because* u *is strictly increasing and continuous in the segment* $I = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$*; the corresponding segment is* $(-\infty, \infty)$*. We denote the inverse that corresponds to this segment by* $y = f(x) = \arctan x$*. From* $y = \arctan x$ *if and only if* $x = \tan y$*, it follows directly that* $\left(\sec^2 y\right) \dfrac{dy}{dx} = 1$ *or* $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$*. On the other hand,* $\tan^2 y + 1 = \sec^2 y$ *with* $x = \tan y$ *implies that* $\sec^2 y = x^2 + 1$*. Therefore,* $\dfrac{dy}{dx} = f'(x) = \dfrac{1}{x^2 + 1}$*.*

Excursion 6.4.10 *Use* $f(x) = \dfrac{x}{1 - x}$ *to verify the Inverse Differentiation Theorem on the segment* $(2, 4)$*; i.e., show that the theorem applies, find the inverse* g *and* *its derivative by the usual algebraic manipulations, and then verify the derivative satisfies equation (6.6)*

Hopefully, you thought to use the Monotonicity Test; that f is strictly increasing in $I = (2, 4)$ follows immediately upon noting that $f'(x) = (1-x)^{-2} > 0$ in I. The corresponding segment $J = \left(-2, -\frac{4}{3}\right)$ is the domain for the inverse g that we seek. The usual algebraic manipulations for finding inverses leads us to solving $x = y(1-y)^{-1}$ for y. Then application of the quotient rule should have led to $g'(x) = (1+x)^{-2}$. Finally, to verify agreement with what is claimed with equation (6.6), substitute g into $f'(x) = (1-x)^{-2}$ and simplify.

6.5 Derivatives of Higher Order

If f is a differentiable function on a set Ω then corresponding to each $x \in \Omega$, there is a uniquely determined $f'(x)$. Consequently, f' is also a function on Ω. We have already seen that f' need not be continuous on Ω. However, if f' is differentiable on a set $\Lambda \subset \Omega$, then its derivative is a function on Λ which can also be considered for differentiability. When they exist, the subsequent derivatives are called higher order derivatives. This process can be continued indefinitely; on the other hand, we could arrive at a function that is not differentiable or, in the case of polynomials, we'll eventually obtain a higher order derivative that is zero everywhere. Note that we can speak of higher order derivatives only after we have isolated the set on which the previous derivative exists.

Definition 6.5.1 *If f is differentiable on a set Ω and f' is differentiable on a set $\Omega_1 \subset \Omega$, then the derivative of f' is denoted by f'' or $\frac{d^2 f}{dx^2}$ and is called the* ***second***

derivative of *f; if the second derivative of f is differentiable on a set $\Omega_2 \subset \Omega_1$, then the derivative of f'', denoted by f''' or $f^{(3)}$ or $\frac{d^3 f}{dx^3}$, is called the* ***third derivative of*** *f. Continuing in this manner, when it exists, $f^{(n)}$ denotes the n^{th} derivative of f and is given by $\left(f^{(n-1)}\right)'$.*

Remark 6.5.2 *The statement "$f^{(k)}$ exists at a point x_0" asserts that $f^{(k-1)}(t)$ is defined in a segment containing x_0 (or in a half-open interval having x_0 as the included endpoint in cases of one-sided differentiability) and differentiable at x_0. If $k > 2$, then the same two claims are true for $f^{(k-2)}$. In general, "$f^{(k)}$ exists at a point x_0" implies that each of $f^{(j)}$, for $j = 1, 2, ..., k-1$, is defined in a segment containing x_0 and is differentiable at x_0.*

Example 6.5.3 *Given $f(x) = \frac{3}{(5+2x)^2}$ in $\mathbb{R} - \left\{-\frac{5}{2}\right\}$, find a general formula for $f^{(n)}$.*

From $f(x) = 3(5+2x)^{-2}$, it follows that $f'(x) = 3\cdot(-2)(5+2x)^{-3}(2)$, $f''(x) = 3\cdot(-2)(-3)(5+2x)^{-4}\left(2^2\right)$, $f^{(3)}(x) = 3\cdot(-2)(-3)(-4)(5+2x)^{-5}\left(2^3\right)$, and $f^{(4)}(x) = 3\cdot(-2)(-3)(-4)(-5)(5+2x)^{-6}\left(2^4\right)$. Basic pattern recognition suggests that

$$f^{(n)}(x) = (-1)^n \cdot 3 \cdot 2^n \cdot (n+1)!\,(5+2x)^{-(n+2)}. \tag{6.7}$$

Remark 6.5.4 *Equation (6.7) was not proved to be the case. While it can be proved by Mathematical Induction, the set-up of the situation is direct enough that claiming the formula from a sufficient number of carefully illustrated cases is sufficient for our purposes.*

Theorem 6.5.5 (Taylor's Approximating Polynomials) *Suppose f is a real function on $[a,b]$ such that there exists $n \in \mathbb{J}$ for which $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists for every $t \in (a,b)$. For $\gamma \in [a,b]$, let*

$$P_{n-1}(\gamma; t) = \sum_{k=0}^{(n-1)} \frac{f^{(k)}(\gamma)}{k!}(t-\gamma)^k.$$

Then, for α and β distinct points in $[a,b]$, there exists a point x between α and β such that

$$f(\beta) = P_{n-1}(\alpha;\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n. \tag{6.8}$$

Excursion 6.5.6 *Fill in what is missing to complete the following proof of Taylor's Approximating Polynomials Theorem.*

Proof. *Since* $P_{n-1}(\alpha;\beta)$, $(\beta-\alpha)^n$ *and* $f(\beta)$ *are fixed, we have that*

$$f(\beta)=P_{n-1}(\alpha;\beta)+M(\beta-\alpha)^n$$

for some $M\in\mathbb{R}$. *Let*

$$g(t)\underset{def}{=}f(t)-P_{n-1}(\alpha;t)-M(t-\alpha)^n.$$

Then g *is a real function on* $[a,b]$ *for which* $\underset{(1)}{\underline{\hspace{5cm}}}$ *is continuous and* $g^{(n)}$ *exists in* (a,b) *because* $\underset{(2)}{\underline{\hspace{12cm}}}$. *From the Properties of Derivatives, for* $t\in(a,b)$, *we have that*

$$g'(t)=f'(t)-\sum_{k=1}^{n-1}\frac{f^{(k)}(\alpha)}{(k-1)!}(t-\alpha)^{k-1}-nM(t-\alpha)^{n-1},$$

and

$$g''(t)=\underset{(3)}{\underline{\hspace{12cm}}}.$$

In general, for j *such that* $1\le j\le(n-1)$ *and* $t\in(a,b)$, *it follows that*

$$g^{(j)}(t)=f^{(j)}(t)-\sum_{k=j}^{n-1}\frac{f^{(k)}(\alpha)}{(k-j)!}(t-\alpha)^{k-j}-\frac{n!}{(n-j)!}M(t-\alpha)^{n-j}.$$

Finally,

$$g^{(n)}(t)=\underset{(4)}{\underline{\hspace{8cm}}}. \tag{6.9}$$

Direct substitution yields that $g(\alpha)=0$. *Furthermore, for each* j, $1\le j\le(n-1)$, $t=\alpha$ *implies that* $\sum_{k=j}^{n-1}\dfrac{f^{(k)}(\alpha)}{(k-j)!}(t-\alpha)^{k-j}=f^{(k)}(\alpha)$; *consequently,*

$$g(\alpha)=g^{(j)}(\alpha)=0 \text{ for each } j,\,1\le j\le(n-1).$$

In view of the choice of M*, we have that* $g(\beta) = 0$*. Because* g *is differentiable in* (a, b)*, continuous in* $[a, b]$*, and* $g(\alpha) = g(\beta) = 0$*, by* $\underset{(5)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}$*, there exists* x_1 *between* α *and* β *such that* $g'(x_1) = 0$*. Assuming that* $n > 1$*, from Rolle's Theorem,* g' *differentiable in* (a, b) *and* $\underset{(6)}{\underline{\qquad\qquad\qquad}}$ *in* $[a, b]$ *with* $g'(\alpha) = g'(x_1) = 0$ *for* $\alpha, x_1 \in (a, b)$ *yields the existence of* x_2 *between* α *and* x_1 *such that* $\underset{(7)}{\underline{\qquad\qquad\qquad}}$*. If* $n > 2$*, Rolle's Theorem can be applied to* g'' *to obtain* x_3 *between* $\underset{(8)}{\underline{\qquad\qquad\qquad}}$ *such that* $g^{(3)}(x_3) = 0$*. We can repeat this process through* $g^{(n)}$*, the last higher order derivative that we are assured exists. After* n *steps, we have that there is an* x_n *between* α *and* x_{n-1} *such that* $g^{(n)}(x_n) = 0$*. Substituting* x_n *into equation (6.9) yields that*

$$0 = g^{(n)}(x_n) = \underset{(9)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}.$$

Hence, there exists a real number x $(= x_n)$ *that is between* α *and* β *such that* $f^{(n)}(x) = n!M$*; i.e.,* $\dfrac{f^{(n)}(x)}{n!} = M$*. The definition of* M *yields equation (6.8).* ■

Acceptable responses are: (1) $g^{(n-1)}$, (2) g is the sum of functions having those properties, (3) $f''(t) - \sum_{k=2}^{n-1} \dfrac{f^{(k)}(\alpha)}{(k-2)!}(t-\alpha)^{k-2} - n(n-1)M(t-\alpha)^{n-2}$, (4) $f^{(n)}(t) - n!M$, (5) Rolle's Theorem or the Mean-Value Theorem, (6) continuous, (7) $g''(x_2) = 0$, (8) α and x_2, and (9) $f^{(n)}(x_n) - n!M$.

Remark 6.5.7 *For* $n = 1$*, Taylor's Approximating Polynomials Theorem is the Mean-Value Theorem. In the general case, the error from using* $P_{n-1}(\alpha; \beta)$ *instead of* $f(\beta)$ *is* $\dfrac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$ *for some* x *between* α *and* β*; consequently, we have an approximation of this error whenever we have bounds on* $\left|f^{(n)}(x)\right|$*.*

Example 6.5.8 *Let* $f(x) = (1-x)^{-1}$ *in* $\left[-\dfrac{3}{4}, \dfrac{7}{8}\right]$*. Then, for each* $n \in \mathbb{J}$*,* $f^{(n)}(x) = n!(1-x)^{-(n+1)}$ *is continuous in* $\left[-\dfrac{3}{4}, \dfrac{7}{8}\right]$*. Consequently, the hypotheses for Tay-*

lor's Approximating Polynomial Theorem are met for each $n \in \mathbb{J}$. For $n = 2$,

$$P_{n-1}(\gamma\,;t) = P_1(\gamma\,;t) = \frac{1}{1-\gamma} + \frac{1}{(1-\gamma)^2}(t-\gamma).$$

If $\alpha = \frac{1}{4}$ and $\beta = -\frac{1}{2}$, the Theorem claims the existence of $x \in \left(-\frac{1}{2}, \frac{1}{4}\right)$ such that

$$f\left(-\frac{1}{2}\right) = P_1\left(\frac{1}{4}; -\frac{1}{2}\right) + \frac{f^{(2)}(x)}{2!}\left(-\frac{1}{2} - \frac{1}{4}\right)^2.$$

Since

$$P_1\left(\frac{1}{4}; -\frac{1}{2}\right) = \frac{1}{1-\frac{1}{4}} + \frac{1}{\left(1-\frac{1}{4}\right)^2}\left(-\frac{1}{2} - \frac{1}{4}\right) = 0$$

we wish to find $x \in \left(-\frac{1}{2}, \frac{1}{4}\right)$ such that $\frac{2}{3} = 0 + \frac{1}{(1-x)^3}\left(\frac{9}{16}\right)$; the only real solution to the last equation is $x_0 = 1 - \frac{3}{2\sqrt[3]{4}}$ which is approximately equal to .055*. Because x_0 is between $\alpha = \frac{1}{4}$ and $\beta = -\frac{1}{2}$, this verifies the Theorem for the specified choices.*

6.6 Differentiation of Vector-Valued Functions

In the case of limits and continuity we have already justified that for functions from $\mathbb{R}$ into $\mathbb{R}^k$, properties are ascribed if and only if the property applies to each coordinate. Consequently, it will come as no surprise that the same "by co-ordinate property assignment" carries over to differentiability.

Definition 6.6.1 *A **vector-valued function** $\mathbf{f}$ from a subset Ω of $\mathbb{R}$ into $\mathbb{R}^k$ is **differentiable at a point** $\zeta \in \Omega$ if and only if $\mathbf{f}$ is defined in a segment containing ζ and there exists an element of $\mathbb{R}^k$, denoted by $\mathbf{f}'(\zeta)$, such that*

$$\lim_{t\to\zeta}\left|\frac{\mathbf{f}(t) - \mathbf{f}(\zeta)}{t-\zeta} - \mathbf{f}'(\zeta)\right| = 0$$

where $|\cdot|$ denotes the Euclidean k-metric.

Lemma 6.6.2 *Suppose that* $f_1, f_2, ..., f_k$ *are real functions on a subset* Ω *of* $\mathbb{R}$ *and* $\mathbf{f}(x) = (f_1(x), f_2(x), ..., f_k(x))$ *for* $x \in \Omega$. *Then* $\mathbf{f}$ *is differentiable at* $\zeta \in \Omega$ *with derivative* $\mathbf{f}'(\zeta)$ *if and only if each of the functions* $f_1, f_2, ..., f_k$ *is differentiable at* ζ *and* $\mathbf{f}'(\zeta) = \left(f_1'(\zeta), f_2'(\zeta), ..., f_k'(\zeta)\right)$.

Proof. For t and ζ in $\mathbb{R}$, we have that

$$\frac{\mathbf{f}(t) - \mathbf{f}(\zeta)}{t - \zeta} - \mathbf{f}'(\zeta) = \left(\frac{f_1(t) - f_1(\zeta)}{t - \zeta} - f_1'(\zeta), ..., \frac{f_k(t) - f_k(\zeta)}{t - \zeta} - f_k'(\zeta)\right).$$

Consequently, the result follows immediately from Lemma 4.3.1 and the Limit of Sequences Characterization for the Limits of Functions. ■

Lemma 6.6.3 *If* $\mathbf{f}$ *is a vector-valued function from* $\Omega \subset \mathbb{R}$ *into* $\mathbb{R}^k$ *that is differentiable at a point* $\zeta \in \Omega$, *then* $\mathbf{f}$ *is continuous at* ζ.

Proof. Suppose that $\mathbf{f}$ is a vector-valued function from $\Omega \subset \mathbb{R}$ into $\mathbb{R}^k$ that is differentiable at a point $\zeta \in \Omega$. Then f is defined in a segment I containing ζ and, for $t \in I$, we have that

$$\begin{aligned}\mathbf{f}(t) - \mathbf{f}(\zeta) &= \left(\frac{f_1(t) - f_1(\zeta)}{t - \zeta}(t - \zeta), ..., \frac{f_k(t) - f_k(\zeta)}{t - \zeta}(t - \zeta)\right) \\ &\longrightarrow \left(f_1'(\zeta) \cdot 0, f_2'(\zeta) \cdot 0, ..., f_k'(\zeta) \cdot 0\right) \text{ as } t \longrightarrow \zeta.\end{aligned}$$

Hence, for each $j \in \mathbb{J}$, $1 \le j \le k$, $\lim_{t \to \zeta} f_j(t) = f_j(\zeta)$; i.e., each f_j is continuous at ζ. From Theorem 5.2.10(a), it follows that $\mathbf{f}$ is continuous at ζ. ■

We note that an alternative approach to proving Lemma 6.6.3 simply uses Lemma 6.6.2. In particular, from Lemma 6.6.2, $\mathbf{f}(x) = (f_1(x), f_2(x), ..., f_k(x))$ differentiable at ζ implies that f_j is differentiable at ζ for each j, $1 \le j \le k$. By Theorem 6.1.13, f_j is continuous at ζ for each j, $1 \le j \le k$, from which Theorem 5.2.10(a) allows us to conclude that $\mathbf{f}(x) = (f_1(x), f_2(x), ..., f_k(x))$ is continuous at ζ.

Lemma 6.6.4 *If* $\mathbf{f}$ *and* $\mathbf{g}$ *are vector-valued functions from* $\Omega \subset \mathbb{R}$ *into* $\mathbb{R}^k$ *that are differentiable at a point* $\zeta \in \Omega$, *then the sum and inner product are also differentiable at* ζ.

Proof. Suppose that $\mathbf{f}(x) = (f_1(x), f_2(x), ..., f_k(x))$ and $\mathbf{g}(x) = (g_1(x), g_2(x), ..., g_k(x))$ are vector-valued functions from $\Omega \subset \mathbb{R}$ into $\mathbb{R}^k$ that are differentiable at a point $\zeta \in \Omega$. Then

$$(\mathbf{f} + \mathbf{g})(x) = ((f_1 + g_1)(x), (f_2 + g_2)(x), ..., (f_k + g_k)(x))$$

and

$$(\mathbf{f} \bullet \mathbf{g})(x) = ((f_1 g_1)(x), (f_2 g_2)(x), ..., (f_k g_k)(x)).$$

From the Properties of Derivatives (c) and (d), for each $j \in \mathbb{J}$, $1 \le j \le k$, $\left(f_j + g_j\right)$ and $\left(f_j g_j\right)$ are differentiable at ζ with $\left(f_j + g_j\right)'(\zeta) = f_j'(\zeta) + g_j'(\zeta)$ and $\left(f_j g_j\right)'(\zeta) = f_j'(\zeta) g_j(\zeta) + f_j(\zeta) g_j'(\zeta)$. From Lemma 6.6.2, it follows that $(\mathbf{f} + \mathbf{g})$ is differentiable at ζ with

$$(\mathbf{f} + \mathbf{g})(\zeta) = \left(\mathbf{f}' + \mathbf{g}'\right)(\zeta) = \left(\left(f_1' + g_1'\right)(\zeta), \left(f_2' + g_2'\right)(\zeta), ..., \left(f_k' + g_k'\right)(\zeta)\right)$$

and $(\mathbf{f} \bullet \mathbf{g})$ is differentiable at ζ with

$$(\mathbf{f} \bullet \mathbf{g})'(\zeta) = \left(\mathbf{f}' \bullet \mathbf{g}\right)(\zeta) + \left(\mathbf{f} \bullet \mathbf{g}'\right)(\zeta).$$

■

The three lemmas might prompt an unwarranted leap to the conclusion that all of the properties that we have found for real-valued differentiable functions on subsets of $\mathbb{R}$ carry over to vector-valued functions on subsets of $\mathbb{R}$. A closer scrutiny reveals that we have not discussed any results for which the hypotheses or conclusions either made use of or relied on the linear ordering on $\mathbb{R}$. Since we loose the existence of a linear ordering when we go to $\mathbb{R}^2$, it shouldn't be a shock that the Mean-Value Theorem does not extend to the vector-valued functions from subsets of $\mathbb{R}$ to $\mathbb{R}^2$.

Example 6.6.5 *For* $x \in \mathbb{R}$, *let* $\mathbf{f}(x) = (\cos x, \sin x)$. *Show that there exists an interval* $[a, b]$ *such that* $\mathbf{f}$ *satisfies the hypotheses of the Mean-Value Theorem without yielding the conclusion.*

From Lemma 6.6.2 and Lemma 6.6.3, we have that $\mathbf{f}$ *is differentiable in* (a, b) *and continuous in* $[a, b]$ *for any* $a, b \in \mathbb{R}$ *such that* $a < b$. *Since* $\mathbf{f}(0) = \mathbf{f}(2\pi) = (1, 0)$, $\mathbf{f}(2\pi) - \mathbf{f}(0) = (0, 0)$. *Because* $\mathbf{f}'(x) = (-\sin x, \cos x)$, $\left|\mathbf{f}'(x)\right| = 1$ *for each* $x \in (0, 2\pi)$. *In particular,* $(\forall x \in (0, 2\pi))\left(\mathbf{f}'(x) \neq (0, 0)\right)$ *from which we see that* $(\forall x \in (0, 2\pi))\left(\mathbf{f}(2\pi) - \mathbf{f}(0) \neq (2\pi - 0)\mathbf{f}'(x)\right)$*; i.e.,*

$$\neg(\exists x)\left[x \in (0, 2\pi) \wedge \left(\mathbf{f}(2\pi) - \mathbf{f}(0) = (2\pi - 0)\mathbf{f}'(x)\right)\right].$$

Remark 6.6.6 *Example 5.18 in our text justifies that L'Hôpital's Rule is also not valid for functions from* $\mathbb{R}$ *into* $\mathbb{C}$.

When we justify that a result known for real-valued differentiable functions on subsets of $\mathbb{R}$ does not carry over to vector-valued functions on subsets of $\mathbb{R}$, it is natural to seek modifications of the original results in terms of properties that might carry over to the different situation. In the case of the Mean-Value Theorem, success in achieved with an inequality that follows directly from the theorem. From the Mean-Value Theorem, if f is a function that is continuous on the interval $I = [a, b]$ and differentiable on the segment $I^\circ = (a, b)$, then there exists a number $\xi \in I^\circ$ such that $f(b) - f(a) = f'(\xi)(b - a)$. Since $\xi \in I^\circ$, $\left|f'(\xi)\right| \le \sup_{x \in I^\circ} \left|f'(x)\right|$. This leads to the weaker statement that $|f(b) - f(a)| \le |b - a| \sup_{x \in I^\circ} \left|f'(x)\right|$. On the other hand, this statement has a natural candidate for generalization because the absolute value or Euclidean 1-metric can be replaced with the Euclidean k-metric. We end this section with a proof of a vector-valued adjustment of the Mean-Value Theorem.

Theorem 6.6.7 *Suppose that* $\mathbf{f}$ *is a continuous mapping of* $[a, b]$ *into* $\mathbb{R}^k$ *that is differentiable in* (a, b). *Then there exists* $x \in (a, b)$ *such that*

$$|\mathbf{f}(b) - \mathbf{f}(a)| \le (b - a)\left|\mathbf{f}'(x)\right| \tag{6.10}$$

Proof. Suppose that $\mathbf{f} = (f_1, f_2)$ is a continuous mapping of $[a, b]$ into $\mathbb{R}^k$ that is differentiable in (a, b) and let $\mathbf{z} = \mathbf{f}(b) - \mathbf{f}(a)$. Equation 6.10 certainly holds if $\mathbf{z} = (\mathbf{0}, \mathbf{0})$; consequently, we suppose that $\mathbf{z} \ne (\mathbf{0}, \mathbf{0})$. By Theorem 5.2.10(b) and Lemma 6.6.4, the real-valued function

$$\phi(t) = \mathbf{z} \bullet \mathbf{f}(t) \ \text{ for } t \in [a, b]$$

is continuous in $[a, b]$ and differentiable in (a, b). Applying the Mean-Value Theorem to ϕ, we have that there exists $x \in (a, b)$ such that

$$\phi(b) - \phi(a) = \phi'(x)(b - a). \tag{6.11}$$

Now,

$$\begin{aligned}
\phi(b) - \phi(a) &= \mathbf{z} \bullet \mathbf{f}(b) - \mathbf{z} \bullet \mathbf{f}(a) \\
&= (\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}(b) - (\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}(a) \\
&= (\mathbf{f}(b) - \mathbf{f}(a)) \bullet (\mathbf{f}(b) - \mathbf{f}(a)) \\
&= \mathbf{z} \bullet \mathbf{z} = |\mathbf{z}|^2 .
\end{aligned}$$

For $z_1 = (f_1(b) - f_1(a))$ and $z_2 = (f_2(b) - f_2(a))$,

$$\begin{aligned} |\phi(x)| &= \left|\mathbf{z} \bullet \mathbf{f}'(x)\right| = \left|z_1 f_1'(x) + z_2 f_2'(x)\right| \\ &\leq \sqrt{|z_1| + |z_2|}\sqrt{\left|f_1'(x)\right| + \left|f_2'(x)\right|} = |\mathbf{z}| \left|\mathbf{f}'(x)\right| \end{aligned}$$

by Schwarz's Inequality. Substituting into equation (6.11) yields

$$|\mathbf{z}|^2 = (b-a)\left|\mathbf{z} \bullet \mathbf{f}'(x)\right| \leq (b-a)\,|\mathbf{z}|\left|\mathbf{f}'(x)\right|$$

which implies $|\mathbf{z}| \leq (b-a)\left|\mathbf{f}'(x)\right|$ because $|\mathbf{z}| \neq 0$. ■

6.7 Problem Set F

1. Use the definition to determine whether or not the given function is differentiable at the specified point. When it is differentiable, give the value of the derivative.
   (a) $f(x) = x^3$; $x = 0$
   (b) $f(x) = \begin{cases} x^3 & \text{, for } 0 \leq x \leq 1 \\ \sqrt{x} & \text{, for } x > 1 \end{cases}$; $x = 1$
   (c) $f(x) = \begin{cases} \sqrt{x}\sin\dfrac{1}{x} & \text{, for } x \neq 0 \\ 0 & \text{, for } x = 0 \end{cases}$; $x = 0$
   (d) $f(x) = \dfrac{9}{2x^2+1}$; $x = 2$
2. Prove that, if f and g are differentiable at ζ, then $F(x) = (f+g)(x)$ is differentiable at ζ and $F'(\zeta) = f'(\zeta) + g'(\zeta)$.
3. Use the definition of the derivative to prove that $f(x) = x^n$ is differentiable on $\mathbb{R}$ for each $n \in \mathbb{J}$.
4. Let $f(x) = \begin{cases} x^2 & \text{, for } x \in \mathbb{Q} \\ 0 & \text{, for } x \notin \mathbb{Q} \end{cases}$.

   Is f differentiable at $x = 0$? Carefully justify your position.

5. If f is differentiable at ζ, prove that

$$\lim_{h\to 0}\frac{f(\zeta+\alpha h)-f(\zeta-\beta h)}{h}=(\alpha+\beta)\,f'(\zeta)\,.$$

6. Discuss the differentiability of the following functions on $\mathbb{R}$.

   (a) $f(x)=|x|+|x+1|$

   (b) $f(x)=x\cdot|x|$

7. Suppose that $f:\mathbb{R}\longrightarrow\mathbb{R}$ is differentiable at a point $c\in\mathbb{R}$. Given any two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $a_n\neq b_n$ for each $n\in\mathbb{J}$ and $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=c$, is it true that

$$\lim_{n\to\infty}\frac{f(b_n)-f(a_n)}{b_n-a_n}=f'(c)?$$

   State your position and carefully justify it.

8. Use the Principle of Mathematical Induction to prove the Leibniz Rule for the n^{th} derivative of a product:

$$(fg)^{(n)}(x)=\sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}(x)\,g^{(k)}(x)$$

   where $\binom{n}{k}=\dfrac{n!}{(n-k)!\,(k!)}$ and $f^{(0)}(x)=f(x)$.

9. Use derivative formulas to find $f'(x)$ for each of the following. Do only the obvious simplifications.

   (a) $f(x)=\dfrac{4x^6+3x-1}{\left(x^5+4x^2\left(5x^3-7x^4\right)^7\right)}$

   (b) $f(x)=\left(4x^2+\dfrac{1+2x}{\left(2+x^2\right)^3}\right)^3\left(4x^9-3x^2+10\right)^2$

   (c) $f(x)=\left(\dfrac{\left(2x^2+3x^5\right)^3+7}{14+\left(4+\sqrt{x^2+3}\right)^4}\right)^{15}$

(d) $f(x) = \left(\left(3x^5 + \frac{1}{x^5}\right)^{10} + \left(4\sqrt{7x^4+3} - 5x^2\right)^5\right)^{12}$

(e) $f(x) = \sqrt{3 + \sqrt{2 + \sqrt{1+x}}}$

10. Complete the following steps to prove that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad and \qquad \lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0.$$

(a) Draw a figure that will serve as an aid towards completion of a proof that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

i. On a copy of a Cartesian coordinate system, draw a circle having radius 1 that is centered at the origin. Then pick an arbitrary point on the part of the circle that is in the first quadrant and label it P.

ii. Label the origin, the point $(1, 0)$, and the point where the line $x = P$ would intersect the x-axis,

$$O, B, \text{ and } A, \text{ respectively.}$$

iii. Suppose that the argument of the point P, in radian measure, is θ. Indicate the coordinates of the point P and show the line segment joining P to A in your diagram.

iv. If your completed diagram is correctly labelled, it should illustrate that

$$\frac{\sin\theta}{\theta} = \frac{\left|\overline{PA}\right|}{\text{length of } \overset{\frown}{PB}}$$

where $\left|\overline{PA}\right|$ denotes the length of the line segment joining points P and A and $\overset{\frown}{PB}$ denotes the arc of the unit circle from the point B to the point P.

v. Finally, the circle having radius $\left|\overline{OA}\right|$ and centered at the origin will pass through the point A and a point and a point on the ray $\overrightarrow{OP}$. Label the point of intersection with $\overrightarrow{OP}$ with the letter C and show the arc $\overset{\frown}{CA}$ on your diagram.

(b) Recall that, for a circle of radius r, the area of a sector subtended by θ radians is given by $\frac{\theta r^2}{2}$. Prove that

$$\frac{\theta\cos^2\theta}{2} < \frac{\cos\theta\sin\theta}{2} < \frac{\theta}{2}$$

for θ satisfying the set-up from part (a).

(c) Prove that $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

(d) Recall that $\sin^2\theta + \cos^2\theta = 1$. Prove that $\lim\limits_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$.

11. The result of Problem 10 in conjunction with the discussion that was offered in the section on Formulas for Derivatives justifies the claim that, for any $x \in \mathbb{R}$, $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, where x is interpreted as radians. Use our Properties of Derivatives and trig identities to prove each of the following.

    (a) $(\tan x)' = \sec^2 x$
    (b) $(\sec x)' = \sec x\tan x$
    (c) $(\csc x)' = -\csc x\cot x$
    (d) $(\ln|\sec x + \tan x|)' = \sec x$
    (e) $(\ln|\csc x - \cot x|)' = \csc x$

12. Use derivative formulas to find $f'(x)$ for each of the following. Do only the obvious simplifications.

    (a) $f(x) = \sin^5\left(3x^4 + \cos^2\left(2x^2 + \sqrt{x^4+7}\right)\right)$
    (b) $f(x) = \dfrac{\tan^3\left(4x+3x^2\right)}{1+\cos^2\left(4x^5\right)}$
    (c) $f(x) = \left(1+\sec^3(3x)\right)^4\left(x^3 + \dfrac{3}{2x^2+1} - \tan x\right)^2$
    (d) $f(x) = \cos^3\left(x^4 - 4\sqrt{1+\sec^4 x}\right)^4$

13. Find each of the following. Use L'Hôpital's Rule when it applies.

    (a) $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\tan x}{x-(\pi/2)}$

    (b) $\lim\limits_{x\to 0} \dfrac{\tan^5 x - \tan^3 x}{1-\cos x}$

    (c) $\lim\limits_{x\to\infty} \dfrac{x^3}{e^{2x}}$

    (d) $\lim\limits_{x\to\infty} \dfrac{4x^3+2x^2-x}{5x^3+3x^2+2x}$

    (e) $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^3}$

    (f) $\lim\limits_{x\to 2^+} (x-2)\ln(x-2)$

14. For $f(x) = x^3$ and $x_0 = 2$ in the Fundamental Lemma of Differentiation, show that $\eta(h) = 6h + h^2$.

15. For $f(x) = \dfrac{x+1}{2x+1}$ and $x_0 = 1$ in the Fundamental Lemma of Differentiation, find the corresponding $\eta(h)$.

16. Suppose that f, g, and h are three real-valued functions on $\mathbb{R}$ and c is a fixed real number such that $f(c) = g(c) = h(c)$ and $f'(c) = g'(c) = h'(c)$. If $\{A_1, A_2, A_3\}$ is a partition of $\mathbb{R}$, and

$$L(x) = \begin{cases} f(x) & \text{, for } x \in A_1 \\ g(x) & \text{, for } x \in A_2 \\ h(x) & \text{, for } x \in A_3 \end{cases},$$

    prove that L is differentiable at $x = c$.

17. If the second derivative for a function f exists at $x_0 \in \mathbb{R}$, show that

$$\lim_{h\to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0).$$

18. For each of the following, find formulas for $f^{(n)}$ in terms of $n \in \mathbb{J}$.

(a) $f(x) = \dfrac{3}{(3x+2)^2}$

(b) $f(x) = \sin(2x)$

(c) $f(x) = \ln(4x+3)$

(d) $f(x) = e^{(5x+7)}$

19. For $f(x) = \begin{cases} e^{-x^{-2}} & , \text{for} \quad x > 0 \\ 0 & , \text{for} \quad x \le 0 \end{cases}$, show that $f^{(n)}(0)$ exists for each $n \in \mathbb{J}$ and is equal to 0.

20. Discuss the monotonicity of each of the following.

(a) $f(x) = x^4 - 4x + 5$

(b) $f(x) = 2x^3 + 3x + 5$

(c) $f(x) = \dfrac{3x+1}{2x-1}$

(d) $f(x) = x^3 e^{-x}$

(e) $f(x) = (1+x)e^{-x}$

(f) $f(x) = \dfrac{\ln x}{x^2}$

21. Suppose that f is a real-valued function on $\mathbb{R}$ for which both the first and second derivatives exist. Determine conditions on f' and f'' that will suffice to justify that the function is increasing at a decreasing rate, increasing at an increasing rate, decreasing at an increasing rate, and decreasing at a decreasing rate.

22. For a function f from a metric space X to a metric space Y, let F_f denote the inverse relation from Y to X. Prove that F_f is a function from rng f into X if and only if f is one-to-one.

23. For each of the following,

- find the segments I_k, $k = 1, 2, ...,$ where f is strictly increasing and strictly decreasing,

- find the corresponding segments $J_k = f(I_k)$ on which the corresponding inverses g_k of f are defined,
- graph f on one Cartesian coordinate system, and each of the corresponding inverses on a separate Cartesian coordinate system, and
- whenever possible, with a reasonable amount of algebraic manipulations, find each g_k.

(a) $f(x) = x^2 + 2x + 2$

(b) $f(x) = \dfrac{2x}{x+2}$

(c) $f(x) = \dfrac{x^2}{2} + 3x - 4$

(d) $f(x) = \sin x$ for $-\dfrac{3\pi}{2} \leq x \leq 2\pi$

(e) $f(x) = \dfrac{2x^3}{3} + x^2 - 4x + 1$

24. Suppose that f and g are strictly increasing in an interval I and that

$$(f - g)(x) > 0$$

for each $x \in I$. Let F and G denote the inverses of f and g, respectively, and J_1 and J_2 denote the respective domains for those inverses. Prove that $F(x) < G(x)$ for each $x \in J_1 \cap J_2$.

25. For each of the following, the Inverse Function Theorem applies on the indicated subset of $\mathbb{R}$. For each given f find the corresponding inverse g. Use the properties of derivatives to find f' and g'. Finally, the formulas for f' and g' to verify equation (6.6).

(a) $f(x) = x^3 + 3x$ for $(-\infty, \infty)$

(b) $f(x) = \dfrac{4x}{x^2+1}$ for $\left(\dfrac{1}{2}, \infty\right)$

(c) $f(x) = e^{4x}$ for $(-\infty, \infty)$

26. For $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{, for } \quad 0 < x \leq 1 \\ 0 & \text{, for } \quad x = 0 \end{cases}$, find the segments I_k, $k-1, 2, \ldots$, where f is strictly increasing and strictly decreasing and the corresponding segments J_k where the Inverse Function Theorem applies.

27. For each of the following, find the Taylor polynomials $P(t)$ as described in Taylor's Approximating Polynomials Theorem about the indicated point γ.

    (a) $f(x) = \dfrac{2}{5-2x}; \gamma = 1$
    (b) $f(x) = \sin x; \gamma = \dfrac{\pi}{4}$
    (c) $f(x) = e^{2x-1}; \gamma = 2$
    (d) $f(x) = \ln(4-x); \gamma = 1$

28. For each of the following functions from $\mathbb{R}$ into $\mathbb{R}^3$, find $\mathbf{f}'$.

    (a) $\mathbf{f}(x) = \left(\dfrac{x^3 \sin x}{x^2+1}, x \tan(3x), e^{2x} \cos(3x-4) \right)$
    (b) $\mathbf{f}(x) = \left(\ln\left(2x^2+3\right), \sec x, \sin^3(2x) \cos^4\left(2+3x^2\right) \right)$

29. For $\mathbf{f}(x) = \left(x^2+2x+2, 3x+2\right)$ in $[0, 2]$, verify equation (6.10).

Chapter 7

Riemann-Stieltjes Integration

Calculus provides us with tools to study nicely behaved phenomena using small discrete increments for information collection. The general idea is to (intelligently) connect information obtained from examination of a phenomenon over a lot of tiny discrete increments of some related quantity to "close in on" or approximate something that behaves in a controlled (i.e., bounded, continuous, etc.) way. The "closing in on" approach is useful only if we can get back to information concerning the phenomena that was originally under study. The benefit of this approach is most beautifully illustrated with the elementary theory of integral calculus over $\mathbb{R}$. It enables us to adapt some "limiting" formulas that relate quantities of physical interest to study more realistic situations involving the quantities.

Consider three formulas that are encountered frequently in most standard physical science and physics classes at the pre-college level:

$$A = l \cdot w \qquad d = r \cdot t \qquad m = d \cdot l.$$

Use the space that is provided to indicate what you "know" about these formulas.

Our use of these formulas is limited to situations where the quantities on the right are constant. The minute that we are given a shape that is not rectangular, a velocity that varies as a function of time, or a density that is determined by our position in (or on) an object, at first, we appear to be "out of luck." However, when the quantities given are well enough behaved, we can obtain bounds on what we wish to study, by making certain assumptions and applying the known formulas incrementally.

Note that except for the units, the formulas are indistinguishable. Consequently, illustrating the “closing in on" or approximating process with any one of them carries over to the others, though the physical interpretation (of course) varies.

Let’s get this more down to earth! Suppose that you build a rocket launcher as part of a physics project. Your launcher fires rockets with an initial velocity of 25 ft/min, and, due to various forces, travels at a rate $v(t)$ given by

$$v(t) = 25 - t^2 \text{ ft/min}$$

where t is the time given in minutes. We want to know how far the rocket travels in the first three minutes after launch. The only formula that we have is $d = r \cdot t$, but to use it, we need a constant rate of speed. *We can make use of the formula to obtain bounds or estimates on the distance travelled.* To do this, we can take increments in the time from 0 minutes to 3 minutes and “pick a relevant rate” to compute a bound on the distance travelled in each section of time. For example, over the entire three minutes, the velocity of the rocket is never more that 25 ft/min.

What does this tell us about the product

$$(25 \text{ ft/min}) \cdot 3 \text{ min}$$

compared to the distance that we seek?

How does the product (16 ft/min) · (3 min) *relate to the distance that we seek?*

We can improve the estimates by taking smaller increments (subintervals of 0 minutes to 3 minutes) and choosing a different “estimating velocity” on each subinterval. For example, using increments of 1.5 minutes and the maximum velocity that is achieved in each subinterval as the estimate for a constant rate through each subinterval, yields an estimate of

$$(25\text{ ft/min})\cdot(1.5\text{ min})+\left(\left(25-\frac{9}{4}\right)\text{ ft/min}\right)\cdot(1.5\text{ min})=\frac{573}{8}\text{ ft.}$$

Excursion 7.0.1 *Find the estimate for the distance travelled taking increments of one minute (which is not small for the purposes of calculus) and using the minimum velocity achieved in each subinterval as the "estimating velocity."*

Hopefully, you obtained 61 feet.

Notice that none of the work done actually gave us the answer to the original problem. Using Calculus, we can develop the appropriate tools to solve the problem as an appropriate limit. This motivates the development of the very important and useful theory of integration. We start with some formal definitions that enable us to carry the "closing in on process" to its logical conclusion.

7.1 Riemann Sums and Integrability

Definition 7.1.1 *Given a closed interval $I=[a,b]$, a* ***partition*** *of I is any finite strictly increasing sequence of points $\mathcal{P}=\{x_0,x_1,\ldots,x_{n-1},x_n\}$ such that $a=x_0$ and $b=x_n$. The* ***mesh of the partition*** *$\{x_0,x_1,\ldots,x_{n-1},x_n\}$ is defined by*

$$\operatorname{mesh}\mathcal{P}=\max_{1\le j\le n}\left(x_j-x_{j-1}\right).$$

Each partition of I, $\{x_0,x_1,\ldots,x_{n-1},x_n\}$, decomposes I into n subintervals $I_j=\left[x_{j-1},x_j\right]$, $j=1,2,\ldots,n$, such that $I_j\cap I_k=x_j$ if and only if $k=j+1$ and is empty for $k\neq j$ or $k\neq(j+1)$. Each such decomposition of I into subintervals is called a ***subdivision of*** **I**.

Notation 7.1.2 *Given a partition $\mathcal{P}=\{x_0,x_1,\ldots,x_{n-1},x_n\}$ of an interval $I=[a,b]$, the two notations Δx_j and $\ell\left(I_j\right)$ will be used for $\left(x_j-x_{j-1}\right)$, the length of the j^{th} subinterval in the partition. The symbol Δ or $\Delta(I)$ will be used to denote an arbitrary subdivision of an interval I.*

If f is a function whose domain contains the closed interval I and f is bounded on the interval I, we know that f has both a least upper bound and a greatest lower bound on I as well as on each interval of any subdivision of I.

Definition 7.1.3 *Given a function f that is bounded and defined on the interval I and a partition $\mathcal{P} = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ of I, let $I_j = \left[x_{j-1}, x_j\right]$, $M_j = \sup\limits_{x\in I_j} f(x)$ and $m_j = \inf\limits_{x\in I_j} f(x)$ for $j = 1, 2, ..., n$. Then the **upper Riemann sum of** f **with respect to the partition** $\mathcal{P}$, denoted by $U(\mathcal{P}, f)$, is defined by*

$$U(\mathcal{P}, f) = \sum_{j=1}^{n} M_j \Delta x_j$$

*and the **lower Riemann sum of** f **with respect to the partition** $\mathcal{P}$, denoted by $L(\mathcal{P}, f)$, is defined by*

$$L(\mathcal{P}, f) = \sum_{j=1}^{n} m_j \Delta x_j$$

where $\Delta x_j = \left(x_j - x_{j-1}\right)$.

Notation 7.1.4 *With the subdivision notation the upper and lower Riemann sums for f are denoted by $U(\Delta, f)$ and $L(\Delta, f)$, respectively.*

Example 7.1.5 *For $f(x) = 2x + 1$ in $I = [0, 1]$ and $\mathcal{P} = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$, $U(\mathcal{P}, f) = \frac{1}{4}\left(\frac{3}{2} + 2 + \frac{5}{2} + 3\right) = \frac{9}{4}$ and $L(\mathcal{P}, f) = \frac{1}{4}\left(1 + \frac{3}{2} + 2 + \frac{5}{2}\right) = \frac{7}{4}$.*

Example 7.1.6 *For $g(x) = \begin{cases} 0 & , \text{for} \quad x \in \mathbb{Q} \cap [0, 2] \\ 1 & , \text{for} \quad x \notin \mathbb{Q} \cap [0, 2] \end{cases}$*
$U(\Delta(I), g) = 2$ and $L(\Delta(I), g) = 0$ for any subdivision of $[0, 2]$.

To build on the motivation that constructed some Riemann sums to estimate a distance travelled, we want to introduce the idea of refining or adding points to partitions in an attempt to obtain better estimates.

Definition 7.1.7 *For a partition* $\mathcal{P}_k = \{x_0, x_1, ..., x_{k-1}, x_k\}$ *of an interval* $I = [a, b]$*, let* Δ_k *denote to corresponding subdivision of* $[a, b]$*. If* $\mathcal{P}_n$ *and* $\mathcal{P}_m$ *are partitions of* $[a, b]$ *having* $n+1$ *and* $m+1$ *points, respectively, and* $\mathcal{P}_n \subset \mathcal{P}_m$*, then* $\mathcal{P}_m$ *is a* ***refinement*** *of* $\mathcal{P}_n$ *or* Δ_m *is a* ***refinement*** *of* Δ_n*. If the partitions* $\mathcal{P}_n$ *and* $\mathcal{P}_m$ *are independently chosen, then the partition* $\mathcal{P}_n \cup \mathcal{P}_m$ *is a* ***common refinement of*** $\mathcal{P}_n$ ***and*** $\mathcal{P}_m$ *and the resulting* $\Delta\left(\mathcal{P}_n \cup \mathcal{P}_m\right)$ *is called a* ***common refinement of*** Δ_n ***and*** Δ_m*.*

Excursion 7.1.8 *Let* $\mathcal{P} = \left\{0, \frac{1}{2}, \frac{3}{4}, 1\right\}$ *and* $\mathcal{P}^* = \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, 1\right\}$.

(a) If Δ *and* Δ^* *are the subdivisions of* $I = [0, 1]$ *that correspond* $\mathcal{P}$ *and* $\mathcal{P}^*$*, respectively, then* $\Delta = \left\{\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right]\right\}$*. Find* Δ^**.*

(b) Set $I_1 = \left[0, \frac{1}{2}\right]$, $I_2 = \left[\frac{1}{2}, \frac{3}{4}\right]$, *and* $I_3 = \left[\frac{3}{4}, 1\right]$*. For* $k = 1, 2, 3$*, let* $\Delta(k)$ *be the subdivision of* I_k *that consists of all the elements of* Δ^* *that are contained in* I_k*. Find* $\Delta(k)$ *for* $k = 1, 2,$ *and* 3.

(c) For $f(x) = x^2$ *and the notation established in parts (a) and (b), find each of the following.*

(i) $m = \inf_{x \in I} f(x)$

(ii) $m_j = \inf_{x \in I_j} f(x)$ *for* $j = 1, 2, 3$

(iii) $m_j^* = \inf \left\{ \inf_{x \in J} f(x) : J \in \Delta(j) \right\}$

(iv) $M = \sup_{x \in I} f(x)$

(v) $M_j = \sup_{x \in I_j} f(x)$ *for* $j = 1, 2, 3$

(vi) $M_j^* = \sup \left\{ \sup_{x \in J} f(x) : J \in \Delta(j) \right\}$

(d) Note how the values m, m_j, m_j^*, M, M_j, *and* M_j^* *compare. What you observed is a special case of the general situation. Let*

$$\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$$

be a partition of an interval $I = [a, b]$, Δ *be the corresponding subdivision of* $[a, b]$ *and* $\mathcal{P}^*$ *denote a refinement of* $\mathcal{P}$ *with corresponding subdivision denoted by* Δ^*. *For* $k = 1, 2, ..., n$, *let* $\Delta(k)$ *be the subdivision of* I_k *consisting of the elements of* Δ^* *that are contained in* I_k. *Justify each of the following claims for any function that is defined and bounded on* I.

(i) If $m = \inf_{x \in I} f(x)$ *and* $m_j = \inf_{x \in I_j} f(x)$, *then, for* $j = 1, 2, ..., n$, $m \leq m_j$ *and* $m_j \leq \inf_{x \in J} f(x)$ *for* $J \in \Delta(j)$.

(ii) If $M = \sup_{x \in I} f(x)$ *and* $M_j = \sup_{x \in I_j} f(x)$*, then, for* $j = 1, 2, ..., n$*,* $M_j \leq M$ *and* $M_j \geq \sup_{x \in J} f(x)$ *for* $J \in \Delta(j)$*.*

Our next result relates the Riemann sums taken over various subdivisions of an interval.

Lemma 7.1.9 *Suppose that* f *is a bounded function with domain* $I = [a, b]$*. Let* Δ *be a subdivision of* I*,* $M = \sup_{x \in I} f(x)$*, and* $m = \inf_{x \in I} f(x)$*. Then*

$$m(b-a) \leq L(\Delta, f) \leq U(\Delta, f) \leq M(b-a) \tag{7.1}$$

and

$$L(\Delta, f) \leq L\left(\Delta^*, f\right) \leq U\left(\Delta^*, f\right) \leq U(\Delta, f) \tag{7.2}$$

for any refinement Δ^* *of* Δ*. Furthermore, if* Δ_γ *and* Δ_λ *are any two subdivisions of* I*, then*

$$L\left(\Delta_\gamma, f\right) \leq U(\Delta_\lambda, f) \tag{7.3}$$

Excursion 7.1.10 *Fill in what is missing to complete the following proofs.*

Proof. Suppose that f is a bounded function with domain $I = [a, b]$, $M = \sup_{x \in I} f(x)$, and $m = \inf_{x \in I} f(x)$. For $\Delta = \{I_k : k = 1, 2, ..., n\}$ an arbitrary subdivision of I, let $M_j = \sup_{x \in I_j} f(x)$ and $m_j = \inf_{x \in I_j} f(x)$. Then $I_j \subset I$ for each $j = 1, 2, ..., n$, we have that

$$m \leq m_j \leq \underset{(1)}{\underline{\hspace{6cm}}}, \text{ for each } j = 1, 2, ..., n.$$

Because $\Delta x_j = \left(x_j - x_{j-1}\right) \geq 0$ for each $j = 1, 2, ..., n$, it follows immediately that

$$\underset{(2)}{\underline{\hspace{3cm}}} = m \sum_{j=1}^{n} \left(x_j - x_{j-1}\right) \leq \sum_{j=1}^{n} m_j \Delta x_j = L(\Delta, f)$$

and

$$\sum_{j=1}^{n} m_j \Delta x_j \leq \sum_{j=1}^{n} M_j \Delta x_j = U(\Delta, f) \leq \underset{(3)}{\underline{\hspace{4cm}}} = M(b-a).$$

Therefore, $m(b-a) \leq L(\Delta, f) \leq U(\Delta, f) \leq M(b-a)$ as claimed in equation (7.1).

Let Δ^* be a refinement of Δ and, for each $k = 1, 2, ..., n$, let $\Delta(k)$ be the subdivision of I_k that consists of all the elements of Δ^* that are contained in I_k. In view of the established conventions for the notation being used, we know that $(\forall J)(J \in \Delta^* \Rightarrow (\exists! k)(k \in \{1, 2, ..., n\} \wedge J \in \Delta(k)))$; also, for each $J \in \Delta(k)$, $J \subset I_k \Rightarrow m_k = \inf\limits_{x \in I_k} f(x) \leq \inf\limits_{x \in J} f(x)$ and $M_k = \sup\limits_{x \in I_k} f(x) \geq \sup\limits_{x \in J} f(x)$. Thus,

$$m_k \ell(I_k) \leq L(\Delta(k), f) \qquad \mathit{and} \qquad M_k \ell(I_k) \geq U(\Delta(k), f)$$

from which it follows that

$$L(\Delta, f) = \sum_{j=1}^{n} m_j \ell(I_j) \leq \sum_{j=1}^{n} L(\Delta(j), f) = L(\Delta^*, f)$$

and

$$U(\Delta, f) = \underset{(4)}{\underline{\hspace{4cm}}} \geq \sum_{j=1}^{n} U(\Delta(j), f) = \underset{(5)}{\underline{\hspace{4cm}}}.$$

From equation (7.1), $L(\Delta^*, f) \leq U(\Delta^*, f)$. Finally, combining the inequalities yields that

$$L(\Delta, f) \leq L(\Delta^*, f) \leq U(\Delta^*, f) \leq U(\Delta, f)$$

which completes the proof of equation (7.2).

Suppose that Δ_γ and Δ_λ are two subdivisions of I. Then $\Delta = \Delta_\gamma \cup \Delta_\lambda$ is $\underset{(6)}{\underline{\hspace{8cm}}}$ Δ_γ and Δ_λ. Because Δ is a refinement of Δ_γ, by the comparison of lower sums given in equation (7.2), $L(\Delta_\gamma, f) \leq L(\Delta, f)$. On the other hand, from Δ being a refinement of Δ_λ, it follows that $\underset{(7)}{\underline{\hspace{6cm}}}$. Combining the inequalities with equation (7.1) leads to equation (7.3). ■

Acceptable responses are: (1) $M_j \leq M$, (2) $m\,(b-a)$, (3) $M\sum_{j=1}^{n}\left(x_j - x_{j-1}\right)$, (4) $\sum_{j=1}^{n} M_j \ell\left(I_j\right)$, (5) $U\left(\Delta^*,\ f\right)$, (6) the common refinement of, and (7) $U\left(\Delta,\ f\right) \leq U\left(f,\ \Delta_\lambda\right)$.\

If f is a bounded function with domain $I = [a,b]$ and $\wp = \wp[a,b]$ is the set of all partitions of $[a,b]$, then the Lemma assures us that $\{L\left(\Delta,\ f\right) : \Delta \in \wp\}$ is bounded above by $(b-a)\sup\limits_{x\in I} f\left(x\right)$ and $\{U\left(\Delta,\ f\right) : \Delta \in \wp\}$ is bounded below by $(b-a)\inf\limits_{x\in I} f\left(x\right)$. Hence, by the least upper bound and greatest lower bound properties of the reals both $\sup\{L\left(\Delta,\ f\right) : \Delta \in \wp\}$ and $\inf\{U\left(\Delta,\ f\right) : \Delta \in \wp\}$ exist; to see that they need not be equal, note that–for the bounded function g given in Example 7.1–we have that $\sup\{L\left(\Delta, g\right) : \Delta \in \wp\} = 0$ while $\inf\{U\left(\Delta, g\right) : \Delta \in \wp\} = 2$.

Definition 7.1.11 *Suppose that f is a function on $\mathbb{R}$ that is defined and bounded on the interval $I = [a,b]$ and $\wp = \wp[a,b]$ is the set of all partitions of $[a,b]$. Then the **upper Riemann integral** and the **lower Riemann integral** are defined by*

$$\overline{\int_a^b} f\left(x\right)dx = \inf_{\mathcal{P}\in\wp} U\left(\mathcal{P},\ f\right) \qquad \text{and} \qquad \underline{\int_a^b} f\left(x\right)dx = \sup_{\mathcal{P}\in\wp} L\left(\mathcal{P},\ f\right),$$

*respectively. If $\overline{\int_a^b} f\left(x\right)dx = \underline{\int_a^b} f\left(x\right)dx$, then f is **Riemann integrable**, or just **integrable**, on I, and the common value of the integral is denoted by $\displaystyle\int_a^b f\left(x\right)dx$.*

Excursion 7.1.12 *Let* $f\left(x\right) = \begin{cases} 5x+3 & ,\textit{for}\quad x \notin \mathbb{Q} \\ 0 & ,\ \textit{for}\quad x \in \mathbb{Q} \end{cases}.$

For each $n \in \mathbb{J}$, let Δ_n denote the subdivision of the interval $[1,2]$ that consists of n segments of equal length. Use $\{\Delta_n : n \in \mathbb{J}\}$ to find an upper bound for

$\overline{\int_1^2} f(x)\,dx$. *[Hint: Recall that* $\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$.*]*

*******Corresponding to each Δ_n you needed to find a useful form for $U(\Delta_n, f)$. Your work should have led you to a sequence for which the limit exists as $n \to \infty$. For $n \in \mathbb{J}$, the partition that gives the desired Δ_n is $\left\{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \ldots, 1+\frac{n}{n}\right\}$. Then $\Delta_n = \{I_1, I_2, \ldots, I_n\}$ with $I_j = \left[1+\frac{j-1}{n}, 1+\frac{j}{n}\right]$ and $M_j = 8+\frac{5j}{n}$ leads to $U(\Delta_n, f) = \frac{21}{2}+\frac{5}{2n}$. Therefore, you should proved that $\overline{\int_1^2} f(x)\,dx \le \frac{21}{2}$.*******

It is a rather short jump from Lemma 7.1.9 to upper and lower bounds on the Riemann integrals. They are given by the next theorem.

Theorem 7.1.13 *Suppose that f is defined on the interval $I = [a, b]$ and $m \le f(x) \le M$ for all $x \in I$. Then*

$$m(b-a) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le M(b-a). \tag{7.4}$$

Furthermore, if f is Riemann integrable on I, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a). \tag{7.5}$$

Proof. Since equation (7.5), is an immediate consequence of the definition of the Riemann integral, we will prove only equation (7.4). Let $\mathcal{D}$ denote the set of all subdivisions of the interval $[a, b]$. By Lemma 7.1.9, we have that, for $\Delta^*, \Delta \in \mathcal{D}$,

$$m(b-a) \le L\left(f, \Delta^*\right) \le U(f, \Delta) \le M(b-a).$$

Since Δ is arbitrary,

$$m(b-a) \leq L\left(f, \Delta^*\right) \leq \inf_{\Delta \in \mathcal{D}} U(f, \Delta)$$

and $\inf\limits_{\Delta \in \mathcal{D}} U(f, \Delta) \leq M(b-a)$; i.e.,

$$m(b-a) \leq L\left(f, \Delta^*\right) \leq \overline{\int_a^b} f(x)\, dx \leq M(b-a).$$

Because Δ^* is also arbitrary, $m(b-a) \leq \sup\limits_{\Delta^* \in \mathcal{D}} L(f, \Delta^*)$ and $\sup\limits_{\Delta^* \in \mathcal{D}} L(f, \Delta^*) \leq \overline{\int_a^b} f(x)\, dx$; i.e.,

$$m(b-a) \leq \underline{\int_a^b} f(x)\, dx \leq \overline{\int_a^b} f(x)\, dx.$$

Combining the inequalities leads to equation (7.4). ■

Before getting into some of the general properties of upper and lower integrals, we are going to make a slight transfer to a more general set-up. A re-examination of the proof of Lemma 7.1.9 reveals that it relied only upon independent application of properties of infimums and supremums in conjunction with the fact that, for any partition $\{x_0, x_1, ..., x_{n-1}, x_n\}$, $x_j - x_{j-1} > 0$ and $\sum\limits_{j=1}^{n}\left(x_j - x_{j-1}\right) = x_n - x_0$. Now, given any function α that is defined and strictly increasing on an interval $[a, b]$, for any partition $\mathcal{P} = \{a = x_0, x_1, ..., x_{n-1}, x_n = b\}$ of $[a, b]$,

$$\alpha(\mathcal{P}) = \{\alpha(a) = \alpha(x_0), \alpha(x_1), ..., \alpha(x_{n-1}), \alpha(x_n) = \alpha(b)\} \subset \alpha([a, b]),$$

$\alpha\left(x_j\right) - \alpha\left(x_{j-1}\right) > 0$ and $\sum\limits_{j=1}^{n}\left(\alpha\left(x_j\right) - \alpha\left(x_{j-1}\right)\right) = \alpha(b) - \alpha(a)$. Consequently, $\alpha(\mathcal{P})$ is a partition of

$$[\alpha(a), \alpha(b)] = \bigcap \{I : I = [c, d] \wedge \alpha(\mathcal{P}) \subset I\},$$

which is the "smallest" interval that contains $\alpha([a, b])$. The case $\alpha(t) = t$ returns us to the set-up for Riemann sums; on the other hand, $\alpha([a, b])$ need not be an interval because α need not be continuous.

Example 7.1.14 *Let* $I = [0,3]$ *and* $\alpha(t) = t^2 + \lfloor t \rfloor$*. Then* $\alpha(I) = [0,1) \cup [2,5) \cup [6,11) \cup \{12\}$*. For the partition* $\mathcal{P} = \left\{0, \frac{1}{2}, 1, \frac{5}{4}, 2, \frac{8}{3}, 3\right\}$ *of* I*,* $\alpha(\mathcal{P}) = \left\{0, \frac{1}{4}, 2, \frac{41}{16}, 6, \frac{82}{9}, 12\right\}$ *is a partition of* $[0,12]$ *which contains* $\alpha(I)$*.*

Definition 7.1.15 *Given a function* f *that is bounded and defined on the closed interval* $I = [a,b]$*, a function* α *that is defined and monotonically increasing on* I*, and a partition* $\mathcal{P} = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ *of* I *with corresponding subdivision* Δ*, let* $M_j = \sup_{x \in I_j} f(x)$ *and* $m_j = \inf_{x \in I_j} f(x)$*, for* $I_j = \left[x_{j-1}, x_j\right]$*. Then the* ***upper Riemann-Stieltjes sum of*** f ***over*** α ***with respect to the partition*** $\mathcal{P}$*, denoted by* $U(\mathcal{P}, f, \alpha)$ *or* $U(\Delta, f, \alpha)$*, is defined by*

$$U(\mathcal{P}, f, \alpha) = \sum_{j=1}^{n} M_j \Delta\alpha_j$$

and the ***lower Riemann-Stieltjes sum of*** f ***over*** α ***with respect to the partition*** $\mathcal{P}$*, denoted by* $L(\mathcal{P}, f, \alpha)$ *or* $L(\Delta, f, \alpha)$*, is defined by*

$$L(\mathcal{P}, f, \alpha) = \sum_{j=1}^{n} m_j \Delta\alpha_j$$

where $\Delta\alpha_j = \left(\alpha\left(x_j\right) - \alpha\left(x_{j-1}\right)\right)$*.*

Replacing x_j with $\alpha\left(x_j\right)$ in the proof of Lemma 7.1.9 and Theorem 7.1.13 yields the analogous results for Riemann-Stieltjes sums.

Lemma 7.1.16 *Suppose that* f *is a bounded function with domain* $I = [a,b]$ *and* α *is a function that is defined and monotonically increasing on* I*. Let* $\mathcal{P}$ *be a partition of* I*,* $M = \sup_{x \in I} f(x)$*, and* $m = \inf_{x \in I} f(x)$*. Then*

$$m(\alpha(b) - \alpha(a)) \le L(\mathcal{P}, f, \alpha) \le U(\mathcal{P}, f, \alpha) \le M(\alpha(b) - \alpha(a)) \tag{7.6}$$

and

$$L(\mathcal{P}, f, \alpha) \le L\left(\mathcal{P}^*, f, \alpha\right) \le U\left(\mathcal{P}^*, f, \alpha\right) \le U(\mathcal{P}, f, \alpha) \tag{7.7}$$

for any refinement $\mathcal{P}^*$ *of* $\mathcal{P}$*. Furthermore, if* Δ_γ *and* Δ_λ *are any two subdivisions of* I*, then*

$$L\left(\Delta_\gamma, f, \alpha\right) \le U(\Delta_\lambda, f, \alpha) \tag{7.8}$$

The bounds given by Lemma 7.1.16 with the greatest lower and least upper bound properties of the reals the following definition.

Definition 7.1.17 *Suppose that f is a function on $\mathbb{R}$ that is defined and bounded on the interval $I = [a, b]$, $\wp = \wp[a, b]$ is the set of all partitions of $[a, b]$, and α is a function that is defined and monotonically increasing on I. Then the* ***upper Riemann-Stieltjes integral*** *and the* ***lower Riemann-Stieltjes integral*** *are defined by*

$$\overline{\int_a^b} f(x)\,d\alpha(x) = \inf_{\mathcal{P}\in\wp} U(\mathcal{P}, f, \alpha) \quad \text{and} \quad \underline{\int_a^b} f(x)\,d\alpha(x) = \sup_{\mathcal{P}\in\wp} L(\mathcal{P}, f, \alpha),$$

respectively. If $\overline{\int_a^b} f(x)\,d\alpha(x) = \underline{\int_a^b} f(x)\,d\alpha(x)$, then f is ***Riemann-Stieltjes integrable****, or* ***integrable with respect to*** *α* ***in the Riemann sense****, on I, and the common value of the integral is denoted by $\displaystyle\int_a^b f(x)\,d\alpha(x)$ or $\displaystyle\int_a^b f\,d\alpha$.*

Definition 7.1.18 *Suppose that α is a function that is defined and monotonically increasing on the interval $I = [a, b]$. Then the set of all functions that are integrable with respect to α in the Riemann sense is denoted by $\Re(\alpha)$.*

Because the proof is essentially the same as what was done for the Riemann upper and lower integrals, we offer the following theorem without proof.

Theorem 7.1.19 *Suppose that f is a bounded function with domain $I = [a, b]$, α is a function that is defined and monotonically increasing on I, and $m \leq f(x) \leq M$ for all $x \in I$. Then*

$$m(\alpha(b) - \alpha(a)) \leq \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha \leq M(\alpha(b) - \alpha(a)). \tag{7.9}$$

Furthermore, if f is Riemann-Stieltjes integrable on I, then

$$m(\alpha(b) - \alpha(a)) \leq \int_a^b f(x)\,d\alpha(x) \leq M(\alpha(b) - \alpha(a)). \tag{7.10}$$

In elementary Calculus, we restricted our study to Riemann integrals of continuous functions. Even there we either glossed over the stringent requirement of needing to check all possible partitions or limited ourselves to functions where some trick could be used. Depending on how rigorous your course was, some examples of finding the integral from the definition might have been based on taking partitions of equal length and using some summation formulas (like was done in Excursion 7.1.12) or might have made use of a special bounding lemma that applied to x^n for each $n \in \mathbb{J}$.

It is not worth our while to grind out some tedious processes in order to show that special functions are integrable. Integrability will only be a useful concept if it is verifiable with a reasonable amount of effort. Towards this end, we want to seek some properties of functions that would guarantee integrability.

Theorem 7.1.20 (Integrability Criterion) *Suppose that f is a function that is bounded on an interval $I = [a, b]$ and α is monotonically increasing on I. Then $f \in \mathfrak{R}(\alpha)$ on I if and only if for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ of I such that*

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \epsilon. \tag{7.11}$$

Excursion 7.1.21 *Fill in what is missing to complete the following proof.*

Proof. Let f be a function that is bounded on an interval $I = [a, b]$ and α be monotonically increasing on I.

Suppose that for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ of I such that

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \epsilon. \tag{*}$$

From the definition of the Riemann-Stieltjes integral and Lemma 7.1.16, we have that

$$L(\mathcal{P}, f, \alpha) \leq \underline{\int_a^b} f(x)\, d\alpha(x) \leq \underset{(1)}{\underline{\hspace{4cm}}} \leq \underset{(2)}{\underline{\hspace{4cm}}}.$$

It follows immediately from (*) that

$$0 \leq \overline{\int_a^b} f(x)\, d\alpha(x) - \underline{\int_a^b} f(x)\, d\alpha(x) < \varepsilon.$$

Since ε was arbitrary and the upper and lower Riemann Stieltjes integrals are constants, we conclude that $\overline{\int_a^b} f(x)\, d\alpha(x) = \underline{\int_a^b} f(x)\, d\alpha(x)$; i.e., $\underset{(3)}{\underline{\hspace{6cm}}}$.

Conversely, suppose that $f \in \Re(\alpha)$ and let $\varepsilon > 0$ be given. For $\wp = \wp[a,b]$ the set of all partitions of $[a,b]$, $\int_a^b f(x)\,d\alpha(x) = \inf_{\mathcal{P}\in\wp} U(\mathcal{P}, f, \alpha)$ and $\int_a^b f(x)\,d\alpha(x) = \sup_{\mathcal{P}\in\wp} L(\mathcal{P}, f, \alpha)$. Thus, $\frac{\varepsilon}{2} > 0$ implies that there exists a $\mathcal{P}_1 \in \wp[a,b]$ such that $\int_a^b f(x)\,d\alpha(x) < U(\mathcal{P}_1, f, \alpha) < \int_a^b f(x)\,d\alpha(x) + \frac{\varepsilon}{2}$ and there exists $\mathcal{P}_2 \in \wp[a,b]$ such that $\int_a^b f(x)\,d\alpha(x) - \frac{\varepsilon}{2}\ \underset{(4)}{\underline{\hspace{7cm}}}$.
Therefore,

$$U(\mathcal{P}_1, f, \alpha) - \int_a^b f(x)\,d\alpha(x) < \frac{\varepsilon}{2} \text{ and } \int_a^b f(x)\,d\alpha(x) - L(\mathcal{P}_2, f, \alpha) < \frac{\varepsilon}{2}. \tag{**}$$

Let $\mathcal{P}$ be the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. Lemma 7.1.16, equation (7.7) applied to (**) yields that

$$\underset{(5)}{\underline{\hspace{3cm}}} - \int_a^b f(x)\,d\alpha(x) < \frac{\varepsilon}{2} \text{ and } \int_a^b f(x)\,d\alpha(x) - \underset{(6)}{\underline{\hspace{3cm}}} < \frac{\varepsilon}{2}.$$

Thus

$$\begin{aligned}
&(U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha)) \\
&\quad = \left(U(\mathcal{P}, f, \alpha) - \int_a^b f(x)\,d\alpha(x)\right) + \left(\underset{(7)}{\underline{\hspace{7cm}}}\right) < \varepsilon.
\end{aligned}$$

■

Acceptable responses are: (1) $\overline{\int_a^b} f(x)\,d\alpha(x)$, (2) $U(\mathcal{P}, f, \alpha)$, (3) $f \in \Re(\alpha)$, (4) $< L(\mathcal{P}_2, f, \alpha) < \int_a^b f(x)\,d\alpha(x)$, (5) $U(\mathcal{P}, f, \alpha)$, (6) $L(\mathcal{P}, f, \alpha)$, and (7) $\int_a^b f(x)\,d\alpha(x) - L(\mathcal{P}, f, \alpha)$.

Theorem 7.1.20 will be useful to us whenever we have a way of closing the gap between functional values on the same intervals. The corollaries give us two “big” classes of integrable functions.

Corollary 7.1.22 *If f is a function that is continuous on the interval $I = [a,b]$, then f is Riemann-Stieltjes integrable on $[a,b]$.*

Proof. Let α be monotonically increasing on I and f be continuous on I. Suppose that $\varepsilon > 0$ is given. Then there exists an $\eta > 0$ such that $[\alpha(b) - \alpha(a)]\,\eta < \varepsilon$. By the Uniform Continuity Theorem, f is uniformly continuous in $[a, b]$ from which it follows that there exists a $\delta > 0$ such that

$$(\forall u)(\forall v)\left[u, v \in I \wedge |u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon\right].$$

Let $\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ be a partition of $[a, b]$ for which mesh $\mathcal{P} < \delta$ and, for each j, $j = 1, 2, ..., n$, set $M_j = \sup\limits_{x_{j-1} \leq x \leq x_j} f(x)$ and $m_j = \inf\limits_{x_{j-1} \leq x \leq x_j} f(x)$. Then $M_j - m_j \leq \eta$ and

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) = \sum_{j=1}^{n} \left(M_j - m_j\right) \Delta\alpha_j \leq \eta \sum_{j=1}^{n} \Delta\alpha_j = \eta\,[\alpha(b) - \alpha(a)] < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have that

$$(\forall \varepsilon)\,(\varepsilon > 0 \Rightarrow (\exists \mathcal{P})\,(\mathcal{P} \in \wp[a, b] \wedge U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon)).$$

In view of the Integrability Criterion, $f \in \Re(\alpha)$. Because α was arbitrary, we conclude that f is Riemann-Stieltjes Integrable (with respect to any monotonically increasing function on $[a, b]$). ■

Corollary 7.1.23 *If f is a function that is monotonic on the interval $I = [a, b]$ and α is continuous and monotonically increasing on I, then $f \in \Re(\alpha)$.*

Proof. Suppose that f is a function that is monotonic on the interval $I = [a, b]$ and α is continuous and monotonically increasing on I. For $\varepsilon > 0$ given, let $n \in \mathbb{J}$, be such that

$$(\alpha(b) - \alpha(a))\,|f(b) - f(a)| < n\varepsilon.$$

Because α is continuous and monotonically increasing, we can choose a partition $\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ of $[a, b]$ such that $\Delta\alpha_j = \left(\alpha\left(x_j\right) - \alpha\left(x_{j-1}\right)\right) = \dfrac{\alpha(b) - \alpha(a)}{n}$. If f is monotonically increasing in I, then, for each $j \in \{1, 2, ..., n\}$,

$M_j = \sup_{x_{j-1} \le x \le x_j} f(x) = f(x_j)$ and $m_j = \inf_{x_{j-1} \le x \le x_j} f(x) = f(x_{j-1})$ and

$$\begin{aligned} U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) &= \sum_{j=1}^{n} \left(M_j - m_j\right) \Delta\alpha_j \\ &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{j=1}^{n} \left(f\left(x_j\right) - f\left(x_{j-1}\right)\right) \\ &= \frac{\alpha(b) - \alpha(a)}{n} \left((f(b) - f(a))\right) < \varepsilon; \end{aligned}$$

while f monotonically decreasing yields that $M_j = f\left(x_{j-1}\right), m_j = f\left(x_j\right)$ and

$$\begin{aligned} U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{j=1}^{n} \left(f\left(x_{j-1}\right) - f\left(x_j\right)\right) \\ &= \frac{\alpha(b) - \alpha(a)}{n} \left((f(a) - f(b))\right) < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have that

$$(\forall \varepsilon)\left(\varepsilon > 0 \Rightarrow (\exists \mathcal{P})\left(\mathcal{P} \in \wp[a,b] \wedge U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon\right)\right).$$

In view of the Integrability Criterion, $f \in \Re(\alpha)$. ■

Corollary 7.1.24 *Suppose that f is bounded on $[a,b]$, f has only finitely many points of discontinuity in $I = [a,b]$, and that the monotonically increasing function α is continuous at each point of discontinuity of f. Then $f \in \Re(\alpha)$.*

Proof. Let $\varepsilon > 0$ be given. Suppose that f is bounded on $[a,b]$ and continuous on $[a,b] - E$ where $E = \{\zeta_1, \zeta_2, ..., \zeta_k\}$ is the nonempty finite set of points of discontinuity of f in $[a,b]$. Suppose further that α is a monotonically increasing function on $[a,b]$ that is continuous at each element of E. Because E is finite and α is continuous at each $\zeta_j \in E$, we can find k pairwise disjoint intervals $\left[u_j, v_j\right]$, $j = 1, 2, ..., k$, such that

$$E \subset \bigcup_{j=1}^{k} \left[u_j, v_j\right] \subsetneq [a,b] \qquad \text{and} \qquad \sum_{j=1}^{k} \left(\alpha\left(v_j\right) - \alpha\left(u_j\right)\right) < \varepsilon^*$$

for any $\varepsilon^* > 0$; furthermore, the intervals can be chosen in such a way that each point $\zeta_m \in E \cap (a, b)$ is an element of the interior of the corresponding interval, $[u_m, v_m]$. Let

$$K = [a, b] - \bigcup_{j=1}^{k} \left(u_j, v_j\right).$$

Then K is compact and f continuous on K implies that f is uniformly continuous there. Thus, corresponding to $\varepsilon^* > 0$, there exists a $\delta > 0$ such that

$$(\forall s)\,(\forall t)\left(s, t \in K \wedge |s - t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon^*\right).$$

Now, let $\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ be a partition of $[a, b]$ satisfying the following conditions:

- $(\forall j)\left(j \in \{1, 2, ..., k\} \Rightarrow u_j \in \mathcal{P} \wedge v_j \in \mathcal{P}\right)$,
- $(\forall j)\left(j \in \{1, 2, ..., k\} \Rightarrow \left(u_j, v_j\right) \cap \mathcal{P} = \emptyset\right)$, and
- $(\forall p)\,(\forall j)\left[\left(p \in \{1, 2, ..., n\} \wedge j \in \{1, 2, ..., k\} \wedge x_{p-1} \neq u_j\right) \Rightarrow \Delta x_p < \delta\right]$.

Note that under the conditions established, $x_{q-1} = u_j$ implies that $x_q = v_j$. If $M = \sup_{x \in I} |f(x)|$, $M_p = \sup_{x_{p-1} \le x \le x_p} f(x)$ and $m_p = \inf_{x_{p-1} \le x \le x_p} f(x)$, then for each p, $M_p - m_p \le 2M$. Furthermore, $M_p - m_p < \varepsilon^*$ as long as $x_{p-1} \neq u_j$. Using commutativity to regroup the summation according to the available bounds yields that

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) = \sum_{j=1}^{n} \left(M_j - m_j\right) \Delta\alpha_j \le [\alpha(b) - \alpha(a)]\,\varepsilon^* + 2M\varepsilon^* < \varepsilon$$

whenever $\varepsilon^* < \dfrac{\varepsilon}{2M + [\alpha(b) - \alpha(a)]}$. Since $\varepsilon > 0$ was arbitrary, from the Integrability Criterion we conclude that $f \in \Re(\alpha)$. ■

Remark 7.1.25 *The three Corollaries correspond to Theorems 6.8, 6.9, and 6.10 in our text.*

As a fairly immediate consequence of Lemma 7.1.16 and the Integrability Criterion we have the following Theorem which is Theorem 6.7 in our text.

Theorem 7.1.26 *Suppose that f is bounded on $[a, b]$ and α is monotonically increasing on $[a, b]$.*

(a) If there exists an $\varepsilon > 0$ and a partition $\mathcal{P}^*$ of $[a, b]$ such that equation (7.11) is satisfied, then equation (7.11) is satisfied for every refinement $\mathcal{P}$ of $\mathcal{P}^*$.

(b) If equation (7.11) is satisfied for the partition $\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ and, for each j, $j = 1, 2, ..., n$, s_j and t_j are arbitrary points in $\left[x_{j-1}, x_j\right]$, then

$$\sum_{j=1}^{n} \left| f\left(s_j\right) - f\left(t_j\right) \right| \Delta\alpha_j < \varepsilon.$$

(c) If $f \in \Re(\alpha)$, equation (7.11) is satisfied for the partition

$$\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$$

and, for each j, $j = 1, 2, ..., n$, t_j is an arbitrary point in $\left[x_{j-1}, x_j\right]$, then

$$\left| \sum_{j=1}^{n} f\left(t_j\right) \Delta\alpha_j - \int_a^b f(x)\, d\alpha(x) \right| < \varepsilon.$$

Remark 7.1.27 *Recall the following definition of Riemann Integrals that you saw in elementary calculus: Given a function f that is defined on an interval $I = \{x : a \leq x \leq b\}$, the "R" sum for $\Delta = \{I_1, I_2, ..., I_n\}$ a subdivision of I is given by*

$$\sum_{j=1}^{j=n} f\left(\xi_j\right) \ell\left(I_j\right)$$

where ξ_j is any element of I_j. The point ξ_j is referred to as a sampling point. To get the "R" integral we want to take the limit over such sums as the mesh of the partitions associated with Δ goes to 0. In particular, if the function f is defined on $I = \{x : a \leq x \leq b\}$ and $\wp[a, b]$ denotes the set of all partitions $\{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ of the interval I, then f is said to be "R" integrable over I if and only if

$$\lim_{mesh \mathcal{P}[a,b] \to 0} \sum_{j=1}^{j=n} f\left(\xi_j\right)\left(x_j - x_{j-1}\right)$$

exists for any choices of $\xi_j \in \left[x_{j-1}, x_j\right]$*. The limit is called the "R" integral and is denoted by* $\int_a^b f(x)\,dx$*.*

Taking $\alpha(t) = t$ *in Theorem 7.1.26 justifies that the old concept of an "R" integrability is equivalent to a Riemann integrability as introduced at the beginning of this chapter.*

The following theorem gives a sufficient condition for the composition of a function with a Riemann-Stieltjes integrable function to be Riemann-Stieltjes integrable.

Theorem 7.1.28 *Suppose* $f \in \Re(\alpha)$ *on* $[a, b]$*,* $m \leq f \leq M$ *on* $[a, b]$*,* ϕ *is continuous on* $[m, M]$*, and* $h(x) = \phi(f(x))$ *for* $x \in [a, b]$*. Then* $h \in \Re(\alpha)$ *on* $[a, b]$*.*

Excursion 7.1.29 *Fill in what is missing in order to complete the proof.*

Proof. For $f \in \Re(\alpha)$ on $[a, b]$ such that $m \leq f \leq M$ on $[a, b]$ and ϕ a continuous function on $[m, M]$, let $h(x) = \phi(f(x))$ for $x \in [a, b]$. Suppose that $\varepsilon > 0$ is given. By the $\underset{(1)}{\underline{\hspace{6cm}}}$, ϕ is uniformly continuous on $[m, M]$. Hence, there exists a $\delta > 0$ such that $\delta < \varepsilon$ and

$$(\forall s)(\forall t)\left(s, t \in [m, M] \wedge |s - t| < \delta \Rightarrow |\phi(s) - \phi(t)| < \varepsilon\right). \qquad (\star)$$

Because $\underset{(2)}{\underline{\hspace{3cm}}}$, there exists a $\mathcal{P} = \{x_0 = a, x_1, ..., x_n = b\} \in \wp[a, b]$ such that

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \delta^2. \qquad (\star\star)$$

For each $j \in \{1, 2, ..., n\}$, let $M_j = \sup_{x_{j-1} \leq x \leq x_j} f(x)$, $m_j = \inf_{x_{j-1} \leq x \leq x_j} f(x)$, $M_j^* = \sup_{x_{j-1} \leq x \leq x_j} h(x)$, and $m_j^* = \inf_{x_{j-1} \leq x \leq x_j} h(x)$. From the Trichotomy Law, we know that

$$A = \left\{j : j \in \{1, 2, ..., n\} \wedge \left(M_j - m_j\right) < \delta\right\}$$

and

$$B = \left\{j : j \in \{1, 2, ..., n\} \wedge \left(M_j - m_j\right) \geq \delta\right\}$$

are disjoint.

If $j \in A$, then $u, v \in \left[x_{j-1}, x_j\right] \Rightarrow |f(u) - f(v)| < \delta$. It follows from $(\star)$ that $\underset{(3)}{\underline{\hspace{3cm}}}$; i.e., $|h(u) - h(v)| < \varepsilon$. Hence, $M_j^* - m_j^* \leq \varepsilon$. Since $B \subset \{1, 2, ..., j\}$, $(\star\star)$ implies that

$$\delta \sum_{j \in B} \Delta\alpha_j \leq \sum_{j \in B} \left(M_j - m_j\right) \Delta\alpha_j \leq \underset{(4)}{\underline{\hspace{6cm}}} < \delta^2.$$

Because $\delta < \varepsilon$ by choice, we conclude that $\sum_{j \in B} \Delta\alpha_j < \varepsilon$. Consequently, for $K = \sup_{m \leq t \leq M} |\phi(t)|$, we have that $\left(M_j^* - m_j^*\right) \leq 2K$ for each $j \in \{1, 2, ..., n\}$ and $\sum_{j \in B} \left(M_j^* - m_j^*\right) \Delta\alpha_j < 2K\varepsilon$. Combining the bounds yields that

$$\begin{aligned} & U(\mathcal{P}, h, \alpha) - L(\mathcal{P}, h, \alpha) \\ = \;& \sum_{j=1}^{n} \left(M_j^* - m_j^*\right) \Delta\alpha_j \\ = \;& \sum_{j \in A} \left(M_j^* - m_j^*\right) \Delta\alpha_j + \sum_{j \in B} \left(M_j^* - m_j^*\right) \Delta\alpha_j \\ \leq \;& \underset{(5)}{\underline{\hspace{3cm}}} + 2K\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the Integrability Criterion allows us to conclude that $h \in \Re(\alpha)$. ■

*******Acceptable responses are: (1) Uniform Continuity Theorem, (2) $f \in \Re(\alpha)$, (3) $|\phi(f(u)) - \phi(f(v))| < \varepsilon$, (4) $U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha)$, (5) $\varepsilon[\alpha(b) - \alpha(a)]$.*******

7.1.1 Properties of Riemann-Stieltjes Integrals

This section offers a list of properties of the various Riemann-Stieltjes integrals. The first lemma allows us to draw conclusions concerning the upper and lower Riemann-Stieltjes sums of a constant times a bounded function in relationship to the upper and lower Riemann-Stieltjes sums of the function.

Lemma 7.1.30 *Suppose that f is a function that is bounded and defined on the interval $I = [a, b]$. For k a nonzero real number and $g = kf$, we have*

$$\inf_{x\in I} g(x) = \begin{cases} k \cdot \inf\limits_{x\in I} f(x) & , \text{if } k > 0 \\ k \cdot \sup\limits_{x\in I} f(x) & , \text{if } k < 0 \end{cases} \qquad \sup_{x\in I} g(x) = \begin{cases} k \cdot \sup\limits_{x\in I} f(x) & , \text{if } k > 0 \\ k \cdot \inf\limits_{x\in I} f(x) & , \text{if } k < 0 \end{cases}.$$

Proof. We will prove two of the four equalities. For f a function that is defined and bounded on the interval $I = [a, b]$ and k a nonzero real number, let $g(x) = kf(x)$.

Suppose that $k > 0$ and that $M = \sup\limits_{x\in I} f(x)$. Then $f(x) \leq M$ for all $x \in I$ and

$$g(x) = kf(x) \leq kM \text{ for all } \; x \in I.$$

Hence, kM is an upper bound for $g(x)$ on the interval I. If kM is not the least upper bound, then there exists an $\varepsilon > 0$ such that $g(x) \leq kM - \varepsilon$ for all $x \in I$. (Here, ε can be taken to be any positive real that is less than or equal to the distance between kM and $\sup\limits_{x\in I} g(x)$.) By substitution, we have $kf(x) \leq kM - \varepsilon$ for all $x \in I$. Since k is positive, the latter is equivalent to

$$f(x) \leq M - \left(\frac{\varepsilon}{k}\right) \text{ for all } \; x \in I$$

which contradicts that M is the supremum of f over I. Therefore,

$$\sup_{x\in I} g(x) = kM = k\sup_{x\in I} f(x).$$

Next, suppose that $k < 0$ and that $M = \sup\limits_{x\in I} f(x)$. Now, $f(x) \leq M$ for all $x \in I$ implies that $g(x) = kf(x) \geq kM$. Hence, kM is a lower bound for $g(x)$ on I. If kM is not a greatest lower bound, then there exists an $\varepsilon > 0$, such that $g(x) \geq kM + \varepsilon$ for all $x \in I$. But, from $kf(x) \geq kM + \varepsilon$ and $k < 0$, we conclude that $f(x) \leq M + (\varepsilon/k)$ for all $x \in I$. Since ε/k is negative, $M + (\varepsilon/k) < M$ which gives us a contradiction to M being the $\sup\limits_{x\in I} f(x)$. Therefore,

$$\inf_{x\in I} g(x) = kM = k\sup_{x\in I} f(x).$$

■

Theorem 7.1.31 (Properties of Upper and Lower Riemann-Stieltjes Integrals) *Suppose that the functions f, f_1, and f_2 are bounded and defined on the closed interval $I = [a, b]$ and α is a function that is defined and monotonically increasing in I.*

(a) *If $g = kf$ for $k \in \mathbb{R} - \{0\}$, then*

$$\underline{\int_a^b} g\,d\alpha = \begin{cases} k\underline{\int_a^b} f(x)\,d\alpha(x) & ,\ if\quad k > 0 \\ k\overline{\int_a^b} f(x)\,d\alpha(x) & ,\ if\quad k < 0 \end{cases}$$

and

$$\overline{\int_a^b} g\,d\alpha = \begin{cases} k\overline{\int_a^b} f(x)\,d\alpha(x) & ,\ if\quad k > 0 \\ k\underline{\int_a^b} f(x)\,d\alpha(x) & ,\ if\quad k < 0 \end{cases}.$$

(b) *If $h = f_1 + f_2$, then*

(i) $\underline{\int_a^b} h(x)\,d\alpha(x) \geq \underline{\int_a^b} f_1(x)\,d\alpha(x) + \underline{\int_a^b} f_2(x)\,d\alpha(x)$, *and*

(ii) $\overline{\int_a^b} h(x)\,d\alpha(x) \leq \overline{\int_a^b} f_1(x)\,d\alpha(x) + \overline{\int_a^b} f_2(x)\,d\alpha(x)$.

(c) *If $f_1(x) \leq f_2(x)$ for all $x \in I$, then*

(i) $\underline{\int_a^b} f_1(x)\,d\alpha(x) \leq \underline{\int_a^b} f_2(x)\,d\alpha(x)$, *and*

(ii) $\overline{\int_a^b} f_1(x)\,d\alpha(x) \leq \overline{\int_a^b} f_2(x)\,d\alpha(x)$.

(d) *If $a < b < c$ and f is bounded on $I^* = \{x : a \leq x \leq c\}$ and α is monotonically increasing on I^*, then*

(i) $\underline{\int_a^c} f(x)\,d\alpha(x) = \underline{\int_a^b} f(x)\,d\alpha(x) + \underline{\int_b^c} f(x)\,d\alpha(x)$, *and*

(ii) $\overline{\int_a^c} f(x)\,d\alpha(x) = \overline{\int_a^b} f(x)\,d\alpha(x) + \overline{\int_b^c} f(x)\,d\alpha(x)$.

Excursion 7.1.32 *Fill in what is missing in order to complete the following proof of part d(i).*

Proof. Suppose that $a < b < c$ and that the function f is bounded in the interval $I^* = [a, c]$. For any finite real numbers γ and λ, let $\mathcal{D}[\gamma, \lambda]$ denote the set of all subdivisions of the interval $[\gamma, \lambda]$. Suppose that $\epsilon > 0$ is given. Since

$$\underline{\int_a^b} f(x)\,d\alpha(x) = \sup_{\Delta\in\mathcal{D}[a,b]} L(\Delta, f, \alpha) \text{ and } \underline{\int_b^c} f(x)\,d\alpha(x) = \sup_{\Delta\in\mathcal{D}[b,c]} L(\Delta, f, \alpha),$$

there exists partitions P_n and P_m of $[a,b]$ and $[b,c]$, respectively, with corresponding subdivisions Δ_n and Δ_m, such that

$$L\left(\Delta_n,\, f, \alpha\right) \geq \underline{\int_a^b} f\left(x\right) d\alpha\left(x\right) - \frac{\epsilon}{2} \text{ and } L\left(\Delta_m,\, f, \alpha\right) \geq \underline{\int_b^c} f\left(x\right) d\alpha\left(x\right) - \frac{\epsilon}{2}.$$

For $P = P_n \cup P_m$, let Δ denote the corresponding subdivision of $[a,c]$. Then

$$\begin{array}{rcl} \underline{\int_a^c} f\left(x\right) d\alpha\left(x\right) & & \\ \geq & \underset{(1)}{\underline{\hspace{3cm}}} & \\ & = & L\left(\Delta_n,\, f, \alpha\right) + L\left(\Delta_m,\, f, \alpha\right) \\ & > & \underset{(2)}{\underline{\hspace{6cm}}} \end{array}$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\underline{\int_a^c} f\left(x\right) d\alpha\left(x\right) \geq \underline{\int_a^b} f\left(x\right) d\alpha\left(x\right) + \underline{\int_b^c} f\left(x\right) d\alpha\left(x\right).$$

Now, we want to show that the inequality can be reversed. Suppose that $\varepsilon > 0$ is given. Since

$$\underline{\int_a^c} f\left(x\right) d\alpha\left(x\right) = \sup_{\Delta \in \mathcal{D}[a,c]} L\left(\Delta,\, f, \alpha\right),$$

There exists a $\Delta' \in \mathcal{D}\left[a,c\right]$ such that

$$L\left(\Delta',\, f, \alpha\right) > \underline{\int_a^c} f\left(x\right) d\alpha\left(x\right) - \varepsilon.$$

For $\mathcal{P}'$ the partition of $[a,c]$ that corresponds to Δ', let $\mathcal{P}'' = \mathcal{P}' \cup \{b\}$ and Δ'' denote the refinement of Δ' that corresponds to $\mathcal{P}''$. From $\underset{(3)}{\underline{\hspace{3cm}}}$

$$L\left(\Delta',\, f, \alpha\right) \leq L\left(\Delta'',\, f, \alpha\right).$$

Because Δ'' is the union of a subdivision of $[a,b]$ and a subdivision of $[b,c]$, it follows from the definition of the lower Riemann-Stieltjes integrals that

$$\underline{\int_a^b} f\left(x\right) d\alpha\left(x\right) + \underline{\int_b^c} f\left(x\right) d\alpha\left(x\right) \geq L\left(\Delta'',\, f, \alpha\right).$$

Therefore,

$$\underline{\int_a^b} f(x)\,d\alpha(x) + \underline{\int_b^c} f(x)\,d\alpha(x) \geq L\left(\Delta'', f, \alpha\right) \geq L\left(\Delta', f, \alpha\right) > \underset{(4)}{\underline{\qquad\qquad\qquad\qquad\qquad}}.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\underline{\int_a^b} f(x)\,d\alpha(x) + \underline{\int_b^c} f(x)\,d\alpha(x) \geq \underline{\int_a^c} f(x)\,d\alpha(x).$$

In view of the Trichotomy Law, $\underline{\int_a^c} f(x)\,d\alpha(x) \geq \underline{\int_a^b} f(x)\,d\alpha(x) + \underline{\int_b^c} f(x)\,d\alpha(x)$ and $\underline{\int_a^b} f(x)\,d\alpha(x) + \underline{\int_b^c} f(x)\,d\alpha(x) \geq \underline{\int_a^c} f(x)\,d\alpha(x)$ yields the desired result. ■
***Acceptable responses include: (1) $L(\Delta, f, \alpha)$,
(2) $\underline{\int_a^b} f(x)\,d\alpha(x) + \underline{\int_b^c} f(x)\,d\alpha(x) - \varepsilon$, (3) Lemma 7.1.16, (4) same as completion for (2).***

Given Riemann-Stieltjes integrable functions, the results of Theorem 7.1.31 translate directly to some of the algebraic properties that are listed in the following Theorem.

Theorem 7.1.33 (Algebraic Properties of Riemann-Stieltjes Integrals) *Suppose that the functions* $f,\ f_1,\ f_2 \in \mathfrak{R}(\alpha)$ *on the interval* $I = [a, b]$.

(a) *If* $g(x) = kf(x)$ *for all* $x \in I$, *then* $g \in \mathfrak{R}(\alpha)$ *and*

$$\int_a^b g(x)\,d\alpha(x) = k \int_a^b f(x)\,d\alpha(x).$$

(b) *If* $h = f_1 + f_2$, *then* $f_1 + f_2 \in \mathfrak{R}(\alpha)$ *and*

$$\int_a^b h(x)\,d\alpha(x) = \int_a^b f_1(x)\,d\alpha(x) + \int_a^b f_2(x)\,d\alpha(x).$$

(c) *If* $f_1(x) \leq f_2(x)$ *for all* $x \in I$, *then*

$$\int_a^b f_1(x)\,d\alpha(x) \leq \int_a^b f_2(x)\,d\alpha(x).$$

(d) *If the function* $f \in \Re(\alpha)$ *also on* $I^* = \{x : b \leq x \leq c\}$, *then* f *is Riemann-Stieltjes integrable on* $I \cup I^*$ *and*

$$\int_a^c f(x)\, d\alpha(x) = \int_a^b f(x)\, d\alpha(x) + \int_b^c f(x)\, d\alpha(x).$$

(e) *If* $|f(x)| \leq M$ *for* $x \in I$, *then*

$$\left| \int_a^b f(x)\, d\alpha(x) \right| \leq M[\alpha(b) - \alpha(a)].$$

(f) *If* $f \in \Re(\alpha^*)$ *on* I, *then* $f \in \Re(\alpha + \alpha^*)$ *and*

$$\int_a^b f d\left(\alpha + \alpha^*\right) = \int_a^b f(x)\, d\alpha(x) + \int_a^b f(x)\, d\alpha^*(x).$$

(g) *If c is any positive real constant, then* $f \in \Re(c\alpha)$ *and*

$$\int_a^b f d(c\alpha) = c\int_a^b f(x)\, d\alpha(x).$$

Remark 7.1.34 *As long as the integrals exist, the formula given in (d) of the Corollary holds regardless of the location of b; i.e., b need not be between a and c.*

Remark 7.1.35 *Since a point has no dimension (that is, has length* 0*), we note that*

$$\int_a^a f(x)\, d\alpha(x) = 0 \text{ for any function } f.$$

Remark 7.1.36 *If we think of the definition of the Riemann-Stieltjes integrals as taking direction into account (for example, with* $\int_{\underline{a}}^{b} f(x)\, d\alpha(x)$ *we had* $a < b$ *and took the sums over subdivisions as we were going from a to b), then it makes sense to introduce the convention that*

$$\int_b^a f(x)\, d\alpha(x) = -\int_a^b f(x)\, d\alpha(x)$$

for Riemann-Stieltjes integrable functions f.

The following result follows directly from the observation that corresponding to each partition of an interval we can derive a partition of any subinterval and vice versa.

Theorem 7.1.37 (Restrictions of Integrable Functions) *If the function f is (Riemann) integrable on an interval I, then $f|_{I^*}$ is integrable on I^* for any subinterval I^* of I.*

Choosing different continuous functions for ϕ in Theorem 7.1.28 in combination with the basic properties of Riemann-Stieltjes integrals allows us to generate a set of Riemann-Stieltjes integrable functions. For example, because $\phi_1(t) = t^2$, $\phi_2(t) = |t|$, and $\phi_3(t) = \gamma t + \lambda$ for any real constants γ and λ, are continuous on $\mathbb{R}$, if $f \in \Re(\alpha)$ on an interval $[a, b]$, then each of $(f)^2$, $|f|$, and $\gamma f + \lambda$ will be Riemann-Stieltjes integrable with respect to α on $[a, b]$. The proof of the next theorem makes nice use of this observation.

Theorem 7.1.38 *If $f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ on $[a, b]$, then $fg \in \Re(\alpha)$.*

Proof. Suppose that $f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ on $[a, b]$. From the Algebraic Properties of the Riemann-Stieltjes Integral, it follows that $(f + g) \in \Re(\alpha)$ on $[a, b]$ and $(f - g) \in \Re(\alpha)$ on $[a, b]$. Taking $\phi(t) = t^2$ in Theorem 7.1.28 yields that $(f + g)^2$ and $(f - g)^2$ are also Riemann-Stieltjes integrable with respect to α on $[a, b]$. Finally, the difference

$$4fg = (f + g)^2 - (f - g)^2 \in \Re(\alpha) \text{ on } [a, b]$$

as claimed. ∎

Theorem 7.1.39 *If $f \in \Re(\alpha)$ on $[a, b]$, then $|f| \in \Re(\alpha)$ and*

$$\left| \int_a^b f(x)\, d\alpha(x) \right| \le \int_a^b |f(x)|\, d\alpha(x).$$

Proof. Suppose $f \in \Re(\alpha)$ on $[a, b]$. Taking $\phi(t) = |t|$ in Theorem 7.1.28 yields that $|f| \in \Re(\alpha)$ on $[a, b]$. Choose $\gamma = 1$, if $\int f(x)\, d\alpha(x) \ge 0$ and $\gamma = -1$, if $\int f(x)\, d\alpha(x) \le 0$. Then

$$\left| \int_a^b f(x)\, d\alpha(x) \right| = \gamma \int_a^b f(x)\, d\alpha(x) \quad \text{and} \quad \gamma f(x) \le |f(x)| \text{ for } x \in [a, b].$$

It follows from Algebraic Properties of the Riemann-Stieltjes Integrals (a) and (c) that

$$\left|\int_a^b f(x)\,d\alpha(x)\right| = \gamma \int_a^b f(x)\,d\alpha(x) = \int_a^b \gamma f(x)\,d\alpha(x) \leq \int_a^b |f(x)|\,d\alpha(x).$$

■

One glaring absence from our discussion has been specific examples of finding the integral for integrable functions using the definition. Think for a moment or so about what the definition requires us to find: First, we need to determine the set of all upper Riemann-Stieltjes sums and the set of all lower Riemann-Stieltjes sums; this is where the subdivisions of the interval over which we are integrating range over all possibilities. We have no uniformity, no simple interpretation for the suprema and infima we need, and no systematic way of knowing when we "have checked enough" subdivisions or sums. On the other hand, whenever we have general conditions that insure integrability, the uniqueness of the least upper and greatest lower bounds allows us to find the value of the integral from considering wisely selected special subsets of the set of all subdivisions of an interval.

The following result offers a sufficient condition under which a Riemann-Stieltjes integral is obtained as a point evaluation. It makes use of the characteristic function. Recall that, for a set S and $A \subset S$, the **characteristic function** $\chi_A : S \to \{0, 1\}$ is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{, if } \quad x \in A \\ 0 & \text{, if } \quad x \in S - A \end{cases}$$

In the following, $\chi_{(0,\infty)}$ denotes the characteristic function with $S = \mathbb{R}$ and $A = (0, \infty)$; i.e.,

$$\chi_{(0,\infty)}(x) = \begin{cases} 1 & \text{, if } \quad x > 0 \\ 0 & \text{, if } \quad x \leq 0 \end{cases}.$$

Lemma 7.1.40 *Suppose that f is bounded on $[a, b]$ and continuous at $s \in (a, b)$. If $\alpha(x) = \chi_{(0,\infty)}(x - s)$, then*

$$\int_a^b f(x)\,d\alpha(x) = f(s).$$

Proof. For each $\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ be an arbitrary partition for $[a, b]$, there exists a $j \in \{1, 2, ..., n\}$ such that $s \in \left[x_{j-1}, x_j\right)$. From the definition of α, we have that $\alpha\left(x_k\right) = 0$ for each $k \in \{1, 2, ..., j-1\}$ and $\alpha\left(x_k\right) = 1$ for each $k \in \{j, ..., n\}$. Hence,

$$\Delta\alpha_k = \alpha\left(x_k\right) - \alpha\left(x_{k-1}\right) = \begin{cases} 1 & , \text{if} \quad k = j \\ \\ 0 & , \text{if} \quad k \in \{1, 2, ..., j-1\} \cup \{j+1, ..., n\} \end{cases},$$

from which we conclude that

$$U\left(\mathcal{P}, f, \alpha\right) = \sup_{x_{j-1} \le x \le x_j} f\left(x\right) \text{ and } L\left(\mathcal{P}, f, \alpha\right) = \inf_{x_{j-1} \le x \le x_j} f\left(x\right).$$

Since f is continuous at s and $\left(x_j - x_{j-1}\right) \le \operatorname{mesh} \mathcal{P}$, $\sup\limits_{x_{j-1} \le x \le x_j} f\left(x\right) \to s$ and $\inf\limits_{x_{j-1} \le x \le x_j} f\left(x\right) \to s$ as $\operatorname{mesh} \mathcal{P} \to 0$. Therefore, $\int_a^b f\left(x\right) d\alpha\left(x\right) = f\left(s\right)$. ■

If the function f is continuous on an interval $[a, b]$, then Lemma 7.1.40 can be extended to a sequence of points in the interval.

Theorem 7.1.41 *Suppose the sequence of nonnegative real numbers* $\{c_n\}_{n=1}^{\infty}$ *is such that* $\sum\limits_{n=1}^{\infty} c_n$ *is convergent,* $\{s_n\}_{n=1}^{\infty}$ *is a sequence of distinct points in* (a, b)*, and* f *is a function that is continuous on* $[a, b]$*. If* $\alpha\left(x\right) = \sum\limits_{n=1}^{\infty} c_n \chi_{(0,\infty)}\left(x - s_n\right)$*, then*

$$\int_a^b f\left(x\right) d\alpha\left(x\right) = \sum_{n=1}^{\infty} c_n f\left(s_n\right).$$

Proof. For $u, v \in (a, b)$ such that $u < v$, let $S_u = \{n \in \mathbb{J} : a < s_n \le u\}$ and $T_v = \{n \in \mathbb{J} : a < s_n \le v\}$. Then

$$\alpha\left(u\right) = \sum_{n \in S_u} c_n \le \sum_{n \in T_v} c_n = \alpha\left(v\right)$$

from which we conclude that α is monotonically increasing. Furthermore, $\alpha\left(a\right) = 0$ and $\alpha\left(b\right) = \sum\limits_{n=1}^{\infty} c_n$.

Let $\varepsilon > 0$ be given. Since $\sum\limits_{n=1}^{\infty} c_n$ is convergent, there exists a positive integer K such that

$$\sum_{n=K+1}^{\infty} c_n < \frac{\varepsilon}{M}$$

where $M = \sup\limits_{x\in[a,b]} |f(x)|$. Let $\alpha_1(x) = \sum\limits_{n=1}^{K} c_n \chi_{(0,\infty)}(x - s_n)$ and $\alpha_2(x) = \sum\limits_{n=K+1}^{\infty} c_n \chi_{(0,\infty)}(x - s_n)$. It follows from Lemma 7.1.40 that

$$\int_a^b f(x)\, d\alpha_1(x) = \sum_{j=1}^{K} c_n f(s_n);$$

while $\alpha_2(b) - \alpha_2(a) < \dfrac{\varepsilon}{M}$ yields that

$$\left|\int_a^b f(x)\, d\alpha_2(x)\right| < \varepsilon.$$

Because $\alpha = \alpha_1 + \alpha_2$, we conclude that

$$\left|\int_a^b f(x)\, d\alpha(x) - \sum_{n=1}^{K} c_n f(s_n)\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\int_a^b f(x)\, d\alpha(x) = \sum_{n=1}^{\infty} c_n f(s_n)$. ■

Theorem 7.1.42 *Suppose that α is a monotonically increasing function such that $\alpha' \in \Re$ on $[a,b]$ and f is a real function that is bounded on $[a,b]$. Then $f \in \Re(\alpha)$ if and only if $f\alpha' \in \Re$. Furthermore,*

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\, \alpha'(x)\, dx.$$

Excursion 7.1.43 *Fill in what is missing in order to complete the following proof of Theorem 7.1.42.*

Proof. Suppose that $\varepsilon > 0$ is given. Since $\alpha' \in \Re$ on $[a, b]$, by the Integrability Criterion, there exists a partition $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ such that

$$U\left(\mathcal{P}, \alpha'\right) - \underset{(1)}{\underline{\hspace{4cm}}} < \frac{\varepsilon}{M} \tag{7.12}$$

where $M = \sup |f(x)|$. Furthermore, from the Mean-Value Theorem, for each $j \in \{1, 2, ..., n\}$ there exists a $t_j \in \left[x_{j-1}, x_j\right]$ such that

$$\Delta\alpha_j = \underset{(2)}{\underline{\hspace{8cm}}} = \alpha'\left(t_j\right)\Delta x_j. \tag{7.13}$$

By Theorem 7.1.26(b) and (7.12), for any $s_j \in \left[x_{j-1}, x_j\right]$, $j \in \{1, 2, ..., n\}$

$$\sum_{j=1}^{n} \left|\alpha'\left(s_j\right) - \alpha'\left(t_j\right)\right| \Delta x_j < \varepsilon. \tag{7.14}$$

With this set-up, we have that

$$\sum_{j=1}^{n} f\left(s_j\right)\Delta\alpha_j = \sum_{j=1}^{n} \underset{(3)}{\underline{\hspace{4cm}}}$$

and

$$\begin{aligned}
&\left|\sum_{j=1}^{n} f\left(s_j\right)\Delta\alpha_j - \sum_{j=1}^{n} f\left(s_j\right)\alpha'\left(s_j\right)\Delta x_j\right| \\
= &\left|\underset{(4)}{\underline{\hspace{8cm}}} - \sum_{j=1}^{n} f\left(s_j\right)\alpha'\left(s_j\right)\Delta x_j\right| \\
= &\left|\sum_{j=1}^{n} f\left(s_j\right)\left[\alpha'\left(t_j\right) - \alpha'\left(s_j\right)\right]\Delta x_j\right| \\
\leq &\; M\left|\sum_{j=1}^{n}\left[\alpha'\left(t_j\right) - \alpha'\left(s_j\right)\right]\Delta x_j\right| < \varepsilon.
\end{aligned}$$

That is,

$$\left|\sum_{j=1}^{n} f\left(s_j\right)\Delta\alpha_j - \sum_{j=1}^{n} f\left(s_j\right)\alpha'\left(s_j\right)\Delta x_j\right| < \varepsilon \tag{7.15}$$

for any choice of points $s_j \in [x_{j-1}, x_j]$, $j = 1, 2, ..., n$. Then

$$\sum_{j=1}^{n} f(s_j)\,\Delta\alpha_j \le U(\mathcal{P}, f\alpha') + \varepsilon$$

and

$$U(\mathcal{P}, f, \alpha) \le U(\mathcal{P}, f\alpha') + \varepsilon.$$

Equation (7.15) also allows us to conclude that

$$U(\mathcal{P}, f\alpha') \le U(\mathcal{P}, f, \alpha) + \varepsilon.$$

Hence,

$$\left|U(\mathcal{P}, f, \alpha) - U(\mathcal{P}, f\alpha')\right| \le \varepsilon. \tag{7.16}$$

Since $\mathcal{P}$ was arbitrary, if follows that (7.16) holds for all $\mathcal{P} \in \wp[a, b]$, the set of all partitions of $[a, b]$. Therefore,

$$\left|\overline{\int_a^b} f(x)\,d\alpha(x) - \underset{(5)}{\underline{\hspace{6cm}}}\right| \le \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, we conclude that, for any function f that is bounded on $[a, b]$,

$$\overline{\int_a^b} f(x)\,d\alpha(x) = \overline{\int_a^b} f(x)\,\alpha'(x)\,dx.$$

Equation (7.15) can be used to draw the same conclusion concerning the comparable lower Riemann and Riemann-Stieltjes integrals in order to claim that

$$\underline{\int_a^b} f(x)\,d\alpha(x) = \underline{\int_a^b} f(x)\,\alpha'(x)\,dx.$$

The combined equalities leads to the desired conclusion. ■
Acceptable responses are: (1) $L(\mathcal{P}, \alpha')$, (2) $\alpha(x_j) - \alpha(x_{j-1})$ (3) $f(s_j)\,\alpha'(t_j)\,\Delta x_j$ (4) $\sum_{j=1}^{n} f(s_j)\,\alpha'(t_j)\,\Delta x_j$, (5) $\overline{\int_a^b} f(x)\,\alpha'(x)\,dx$.

Recall that our original motivation for introducing the concept of the Riemann integral was adapting formulas such as $A = l \cdot w$, $d = r \cdot t$ and $m = d \cdot l$ to more general situations; the Riemann integral allow us to replace one of the "constant dimensions" with functions that are at least bounded where being considered. The Riemann-Stieltjes integral allows us to replace both of the "constant dimensions" with functions. Remark 6.18 on page 132 of our text describes a specific example that illustrates to flexibility that has been obtained.

The last result of this section gives us conditions under which we can transfer from one Riemann-Stieltjes integral set-up to another one.

Theorem 7.1.44 (Change of Variables) *Suppose that* ϕ *is a strictly increasing continuous function that maps an interval* $[A, B]$ *onto* $[a, b]$*,* α *is monotonically increasing on* $[a, b]$*, and* $f \in \Re(\alpha)$ *on* $[a, b]$*. For* $y \in [A, B]$*, let* $\beta(y) = \alpha(\phi(y))$ *and* $g(y) = f(\phi(y))$*. Then* $g \in \Re(\beta)$ *and*

$$\int_A^B g(y)\,d\beta(y) = \int_a^b f(x)\,d\alpha(x).$$

Proof. Because ϕ is strictly increasing and continuous, each partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\} \in \wp[a, b]$ if and only if $\mathcal{Q} = \{y_0, y_1, \ldots, y_n\} \in \wp[A, B]$ where $x_j = \phi\left(y_j\right)$ for each $j \in \{0, 1, \ldots, n\}$. Since $f\left(\left[x_{j-1}, x_j\right]\right) = g\left(\left[y_{j-1}, y_j\right]\right)$ for each j, it follows that

$$U(\mathcal{Q}, g, \beta) = U(\mathcal{P}, f, \alpha) \qquad and \qquad L(\mathcal{Q}, g, \beta) = L(\mathcal{P}, f, \alpha).$$

The result follows immediately from the Integrability Criterion. ■

7.2 Riemann Integrals and Differentiation

When we restrict ourselves to Riemann integrals, we have some nice results that allow us to make use of our knowledge of derivatives to compute integrals. The first result is both of general interest and a useful tool for proving some to the properties that we seek.

Theorem 7.2.1 (Mean-Value Theorem for Integrals) *Suppose that* f *is continuous on* $I = [a, b]$*. Then there exists a number* ξ *in* I *such that*

$$\int_a^b f(x)\,dx = f(\xi)(b - a).$$

Proof. This result follows directly from the bounds on integrals given by Theorem 7.1.13 and the Intermediate Value Theorem. Since f is continuous on $[a, b]$, it is integrable there and, by Theorem 7.1.13,

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

where $m = \inf_{x\in I} f(x) = \min_{x\in I} f(x) = f(x_0)$ for some $x_0 \in I$ and $M = \sup_{x\in I} f(x) = \max_{x\in I} f(x) = f(x_1)$ for some $x_1 \in I$. Now, $A = \dfrac{\int_a^b f(x)\,dx}{(b-a)}$ is a real number such that $m \le A \le M$. By the Intermediate Value Theorem, $f(x_0) \le A \le f(x_1)$ implies that there exists a $\xi \in I$ such that $f(\xi) = A$. ■

The following two theorems are two of the most celebrated results from integral calculus. They draw a clear and important connection between integral calculus and differential calculus. The first one makes use of the fact that integrability on an interval allows us to define a new function in terms of the integral. Namely, if f is Riemann integrable on the interval $[a, b]$, then, by the Theorem on Restrictions of Integrable Functions, we know that it is integrable on every subinterval of $[a, b]$. In particular, for each $x \in [a, b]$, we can consider

$$f : x \longrightarrow \int_a^x f(t)\,dt.$$

This function is sometimes referred to as the accumulation of f–probably as a natural consequence of relating the process of integration to finding the area between the graph of a positive function and the real axis. The variable t is used as the dummy variable because x is the argument of the function. The accumulation function is precisely the object that will allow us to relate the process of integration back to differentiation at a point.

Theorem 7.2.2 (The First Fundamental Theorem of Calculus) *Suppose that* $f \in \Re$ *on* $I = [a, b]$*. Then the function* F *given by*

$$F(x) = \int_a^x f(t)\,dt$$

is uniformly continuous on $[a, b]$*. If* f *is continuous on* I*, then* F *is differentiable in* (a, b) *and, for each* $x \in (a, b)$*,* $F'(x) = f(x)$*.*

Proof. Suppose that $u, v \in [a, b]$. Without loss of generality we can assume that $u < v$. Then, from the Algebraic Properties of Riemann-Stieltjes Integrals (d) and (e),

$$|F(v) - F(u)| = \left| \int_u^v f(t)\, dt \right| \leq M |u - v|$$

where $M = \sup\limits_{t \in I} |f(t)|$. Thus,

$$(\forall \varepsilon > 0)\, (\forall u)\, (\forall v) \left(u, v \in I \wedge |u - v| < \delta = \frac{\varepsilon}{M} \Rightarrow |F(v) - F(u)| < \varepsilon \right); \text{i.e.,}$$

F is uniformly continuous on I.

For the second part, suppose f is continuous on $[a, b]$ and $x \in (a, b)$. Then there exists δ_1 such that $\{x + h : |h| < \delta_1\} \subset (a, b)$. Since f is continuous it is integrable on every subinterval of I, for $|h| < \delta_1$, we have that each of $\int_a^{x+h} f(t)\, dt$, $\int_a^x f(t)\, dt$, and $\int_x^{x+h} f(t)\, dt$ exists and

$$\int_a^{x+h} f(t)\, dt = \int_a^x f(t)\, dt + \int_x^{x+h} f(t)\, dt.$$

Consequently, for any h, with $|h| < \delta_1$, we have that

$$F(x + h) - F(x) = \int_x^{x+h} f(t)\, dt.$$

By the Mean-Value Theorem for Integrals, there exists ξ_h with $|x - \xi_h| < \delta_1$ such that

$$\int_x^{x+h} f(t)\, dt = f(\xi_h) \cdot h.$$

Hence, for $|h| < \delta_1$,

$$\frac{F(x + h) - F(x)}{h} = f(\xi_h)$$

where $|x - \xi_h| < \delta_1$. Now, suppose that $\epsilon > 0$ is given. Since f is continuous at x, there exists a $\delta_2 > 0$ such that $|f(w) - f(x)| < \epsilon$ whenever $|w - x| < \delta_2$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then, for $|h| < \delta$, we have

$$\left| \frac{F(x + h) - F(x)}{h} - f(x) \right| = |f(\xi_h) - f(x)| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)\text{; i.e.,}$$

$F'(x) = f(x)$. Since $x \in (a, b)$ was arbitrary, we conclude that F is differentiable on the open interval (a, b). ■

Theorem 7.2.3 (The Second Fundamental Theorem of Calculus) *If $f \in \Re$ on $I = [a, b]$ and there exists a function F that is differentiable on $[a, b]$ with $F' = f$, then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Excursion 7.2.4 *Fill in what is missing in order to complete the following proof.*

Proof. Suppose that $\varepsilon > 0$ is given. For $f \in \Re$, by the $\underset{(1)}{\underline{\hspace{3cm}}}$, we can choose a partition $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ such that $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon$. By the Mean-Value Theorem, for each $j \in \{1, 2, ..., n\}$ there is a point $t_j \in [x_{j-1}, x_j]$ such that

$$F(x_j) - F(x_{j-1}) = F'(t_j)\,\Delta x_j = \underset{(2)}{\underline{\hspace{3cm}}}.$$

Hence,

$$\sum_{j=1}^{n} f(t_j)\,\Delta x_j = \underset{(3)}{\underline{\hspace{6cm}}}$$

On the other hand, from Theorem 7.1.26(c), $\left| \sum_{j=1}^{n} f(t_j)\,\Delta x_j - \int_a^b f(x)\,dx \right| < \varepsilon$. Therefore,

$$\left| \underset{(3)}{\underline{\hspace{6cm}}} - \int_a^b f(x)\,dx \right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\int_a^b f(t)\,dt = F(b) - F(a)$. ■

Acceptable responses are: (1) Integrability Criterion, (2) $f(t_j)\,\Delta x_j$, (3) $F(b) - F(a)$.

Remark 7.2.5 *The statement of the First Fundamental Theorem of Calculus differs from the one that you had in elementary Calculus. If instead of taking* f *to be integrable in* $I = [a, b]$*, we take* f *to be integrable on an open interval containing* I*, we can claim that* $G(x) = \int_a^x f(t)\,dt$ *is differentiable on* $[a, b]$ *with* $G'(x) = f(x)$ *on* $[a, b]$*. This enables us to offer a slightly different proof for the Second Fundamental Theorem of Calculus. Namely, it follows that if* F *is any antiderivative for* f *then* $F - G = c$ *for some constant* c *and we have that*

$$F(b) - F(a) = [G(b) + c] - [G(a) + c] = [G(b) - G(a)] = \int_a^b f(t)\,dt.$$

Remark 7.2.6 *The Fundamental Theorems of Calculus give us a circumstance under which finding the integral of a function is equivalent to finding a primitive or antiderivative of a function. When* f *is a continuous function, we conclude that it has a primitive and denote the set of all primitives by* $\int f(x)\,dx$*; to find the definite integral* $\int_a^b f(x)\,dx$*, we find any primitive of* f*,* F*, and we conclude that*

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

7.2.1 Some Methods of Integration

We illustrate with two methods, namely substitution and integration-by-parts. The theoretical foundation for the method of substitution is given by Theorem 7.1.42 and the Change of Variables Theorem.

Theorem 7.2.7 *Suppose that the function* f *is continuous on a segment* I*, the functions* u *and* $\dfrac{du}{dx}$ *are continuous on a segment* J*, and the range of* u *is contained in* I*. If* $a, b \in J$*, then*

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

Proof. By the First Fundamental Theorem of Calculus, for each $c \in I$, the function

$$F(u) = \int_c^u f(t)\,dt$$

is differentiable with $F'(u) = f(u)$ for $u \in I$ with $c \leq u$. By the Chain Rule, if $G(x) = F(u(x))$, then $G'(x) = F'(u(x))\, u'(x)$. Hence, $G'(x) = f(u(x))\, u'(x)$. Also, G' is continuous from the continuity of f, u, and u'. It follows from the Second Fundamental Theorem of Calculus and the definition of G that

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_a^b G'(x)\, dx = G(b) - G(a) = F(u(b)) - F(u(a)).$$

From the definition of F, we conclude that

$$F(u(b)) - F(u(a)) = \int_c^{u(b)} f(t)\, dt - \int_c^{u(a)} f(t)\, dt = \int_{u(a)}^{u(b)} f(t)\, dt.$$

The theorem follows from the transitivity of equals. ■

Example 7.2.8 *Use of the Substitution Method of Integration to find*

$$\int_0^{\pi/4} \cos^2\left(3t + \frac{\pi}{4}\right) \sin\left(3t + \frac{\pi}{4}\right) dt$$

(a) Take $u = \cos\left(3t + \frac{\pi}{4}\right)$

(b) Take $u = \cos^3\left(3t + \frac{\pi}{4}\right)$.

The other common method of integration with which we should all be familiar is known as Integration by Parts. The IBP identity is given by

$$\int u\, dv = uv - \int v\, du$$

for u and v differentiable and follows from observing that $d(uv) = udv + vdu$, which is the product rule in differential notation. This enables us to tackle integrands that "are products of functions not related by differentiation" and some special integrands, such as the inverse trig functions.

Example 7.2.9 *Examples of the use of the Integration-by-Parts method of integration.*

1. *Find* $\int x^3 \cdot \sqrt{1-x^2}dx$.

2. *Find* $\int \arctan(x)\, dx$

3. *Find* $\int e^{2x}\sin(3x)\,dx$

7.2.2 The Natural Logarithm Function

The Fundamental Theorems enable us to find integrals by looking for antiderivatives. The formula $(x^n)' = nx^{n-1}$ for n an integer leads us to conclude that $\int x^k dx = \frac{x^{k+1}}{k+1} + C$ for any constant C as long as $k+1 \neq 0$. So we can't use a simple formula to determine

$$\int_a^b \frac{1}{x}dx,$$

though we know that it exists for any finite closed interval that does not contain 0 because x^{-1} is continuous in any such interval. This motivates us to introduce a notation for a simple form of this integral.

Definition 7.2.10 *The* ***natural logarithm function****, denoted by* ln *, is defined by the formula*

$$\ln x = \int_1^x \frac{1}{t}dt, \text{ for every } x > 0.$$

As fairly immediate consequences of the definition, we have the following Properties of the Natural Logarithm Function. Let $f(x) = \ln x$ for $x > 0$ and suppose that a and b are positive real numbers. Then, the following properties hold:

1. $\ln(ab) = \ln(a) + \ln(b)$,
2. $\ln(a/b) = \ln(a) - \ln(b)$,
3. $\ln(1) = 0$,

4. $\ln(a^r) = r \cdot \ln(a)$ for every rational number r,
5. $f'(x) = \dfrac{1}{x}$,
6. f is increasing and continuous on $I = \{x : 0 < x < +\infty\}$,
7. $\dfrac{1}{2} \le \ln(2) \le 1$,
8. $\ln\ x \longrightarrow +\infty$ as $x \to +\infty$,
9. $\ln\ x \longrightarrow -\infty$ as $x \to 0^+$, and
10. the range of f is all of $\mathbb{R}$.

Remark 7.2.11 *Once we have property (6), the Inverse Function Theorem guarantees the existence of an inverse function for* $\ln x$*. This leads us back to the function* e^x.

7.3 Integration of Vector-Valued Functions

Building on the way that limits, continuity, and differentiability from single-valued functions translated to vector-valued functions, we define Riemann-Stieltjes integrability of vector-valued functions by assignment of that property to the coordinates.

Definition 7.3.1 *Given a vector-valued (n-valued) function* $\mathbf{f} = (f_1, f_2, ..., f_n)$ *from* $[a, b]$ *into* $\mathbb{R}^n$ *where the real-valued functions* $f_1, f_2, ..., f_n$ *are bounded on the interval* $I = [a, b]$ *and a function* α *that is defined and monotonically increasing on* I*,* $\mathbf{f}$ *is* ***Riemann-Stieltjes integrable with respect to*** α *on* I*, written* $\mathbf{f} \in \Re(\alpha)$*, if and only if* $(\forall j)\left(j \in \{1, 2, ..., n\} \Rightarrow f_j \in \Re(\alpha)\right)$. *In this case,*

$$\int_a^b \mathbf{f}(x)\, d\alpha(x) \underset{def}{=} \left(\int_a^b f_1(x)\, d\alpha(x), \int_a^b f_2(x)\, d\alpha(x), ..., \int_a^b f_n(x)\, d\alpha(x)\right).$$

Because of the nature of the definition, any results for Riemann-Stieltjes integrals that involved “simple algebraic evaluations” can be translated to the vector-valued case.

Theorem 7.3.2 *Suppose that the vector-valued functions* $\mathbf{f}$ *and* $\mathbf{g}$ *are Riemann-Stieltjes integrable with respect to* α *on the interval* $I = [a, b]$.

(a) If k *is a real constant, then* $k\mathbf{f} \in \mathfrak{R}(\alpha)$ *on* I *and*

$$\int_a^b k\mathbf{f}(x)\, d\alpha(x) = k \int_a^b \mathbf{f}(x)\, d\alpha(x).$$

(b) If $\mathbf{h} = \mathbf{f} + \mathbf{g}$*, then* $\mathbf{h} \in \mathfrak{R}(\alpha)$ *and*

$$\int_a^b \mathbf{h}(x)\, d\alpha(x) = \int_a^b \mathbf{f}(x)\, d\alpha(x) + \int_a^b \mathbf{g}(x)\, d\alpha(x).$$

(c) If the function $f \in \mathfrak{R}(\alpha)$ *also on* $I^* = \{x : b \le x \le c\}$*, then* f *is Riemann-Stieltjes integrable with respect to* α *on* $I \cup I^*$ *and*

$$\int_a^c \mathbf{f}(x)\, d\alpha(x) = \int_a^b \mathbf{f}(x)\, d\alpha(x) + \int_b^c \mathbf{f}(x)\, d\alpha(x).$$

(d) If $\mathbf{f} \in \mathfrak{R}(\alpha^*)$ *on* I*, then* $\mathbf{f} \in \mathfrak{R}(\alpha + \alpha^*)$ *and*

$$\int_a^b \mathbf{f} d\left(\alpha + \alpha^*\right) = \int_a^b \mathbf{f}(x)\, d\alpha(x) + \int_a^b \mathbf{f}(x)\, d\alpha^*(x).$$

(e) If c *is any positive real constant, then* $\mathbf{f} \in \mathfrak{R}(c\alpha)$ *and*

$$\int_a^b \mathbf{f} d(c\alpha) = c \int_a^b \mathbf{f}(x)\, d\alpha(x).$$

Theorem 7.3.3 *Suppose that* α *is a monotonically increasing function such that* $\alpha' \in \mathfrak{R}$ *on* $[a, b]$ *and* $\mathbf{f}$ *is a vector-valued function that is bounded on* $[a, b]$*. Then* $\mathbf{f} \in \mathfrak{R}(\alpha)$ *if and only if* $\mathbf{f}\alpha' \in \mathfrak{R}$*. Furthermore,*

$$\int_a^b \mathbf{f}(x)\, d\alpha(x) = \int_a^b \mathbf{f}(x)\, \alpha'(x)\, dx.$$

Theorem 7.3.4 *Suppose that* $\mathbf{f} = (f_1, f_2, ..., f_n) \in \mathfrak{R}$ *on* $I = [a, b]$.

(a) *Then the vector-valued function* $\mathbf{F}$ *given by*

$$\mathbf{F}(x) = \left(\int_a^x f_1(t)\,dt, \int_a^x f_2(t)\,dt, \ldots, \int_a^x f_n(t)\,dt\right) \text{ for } x \in I$$

is continuous on $[a, b]$. *Furthermore, if* $\mathbf{f}$ *is continuous on* I, *then* $\mathbf{F}$ *is differentiable in* (a, b) *and, for each* $x \in (a, b)$, $\mathbf{F}'(x) = \mathbf{f}(x) = (f_1(x), f_2(x), \ldots, f_n(x))$.

(b) *If there exists a vector-valued function* $\mathbf{G}$ *on* I *that is differentiable there with* $\mathbf{G}' = \mathbf{f}$, *then*

$$\int_a^b \mathbf{f}(t)\,dt = \mathbf{G}(b) - \mathbf{G}(a).$$

On the other hand, any of the results for Riemann-Stieltjes integrals of real-valued functions that involved inequalities require independent consideration for formulations that might apply to the vector-valued situation; while we will not pursue the possibilities here, sometimes other geometric conditions can lead to analogous results. The one place where we do have an almost immediate carry over is with Theorem 7.1.39 because the inequality involved the absolute value which generalizes naturally to an inequality in terms of the Euclidean metric. The generalization–natural as it is–still requires proof.

Theorem 7.3.5 *Suppose that* $\mathbf{f} : [a, b] \to \mathbb{R}^n$ *and* $\mathbf{f} \in \mathfrak{R}(\alpha)$ *on* $[a, b]$ *for some* α *that is defined and monotonically increasing on* $[a, b]$. *Then* $|\mathbf{f}| \in \mathfrak{R}(\alpha)$ *and*

$$\left|\int_a^b \mathbf{f}(x)\,d\alpha(x)\right| \le \int_a^b |\mathbf{f}(x)|\,d\alpha(x). \tag{7.17}$$

Excursion 7.3.6 *Fill in what is missing in order to complete the following proof.*

Proof. Suppose that $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in \mathfrak{R}(\alpha)$ on $I = [a, b]$. Then

$$|\mathbf{f}(x)| = \sqrt{f_1^2(x) + f_2^2(x) + \ldots + f_n^2(x)} \ge 0 \text{ for } x \in I \tag{7.18}$$

and, because $\mathbf{f}$ is $\underset{(1)}{\underline{\qquad\qquad}}$ on I, there exists $M > 0$ such that $|\mathbf{f}(I)| = \{|\mathbf{f}(x)| : x \in I\} \subset [0, M]$. Since $(\forall j)\left(j \in \{1, 2, \ldots, n\} \Rightarrow f_j \in \mathfrak{R}(\alpha)\right)$ and the function $\phi(t) = t^2$ is continuous on $\mathbb{R}$, by $\underset{(2)}{\underline{\hspace{3cm}}}$, $(\forall j)\left(j \in \{1, 2, ..., n\} \Rightarrow f_j^2 \in \Re(\alpha)\right)$. From Algebraic Property (b) of the Riemann-Stieltjes integral, $f_1^2(x) + f_2^2(x) + ... + f_n^2(x) \in \Re(\alpha)$. Taking $\phi^*(t) = \sqrt{t}$ for $t \geq 0$ in Theorem 7.1.28 yields that $|\mathbf{f}|$ $\underset{(3)}{\underline{\hspace{3cm}}}$.

Since (7.17) is certainly satisfied if $\mathbf{f}$, we assume that $\mathbf{f} \neq \mathbf{0}$. For each $j \in \{1, 2, ..., n\}$, let $w_j = \int_a^b f_j(x)\,d\alpha(x)$ and set $\mathbf{w} = \int_a^b \mathbf{f}(x)\,d\alpha(x)$. Then

$$|\mathbf{w}|^2 = \sum_{j=1}^{n} w_j^2 = \sum_{j=1}^{n} w_j \int_a^b f_j(t)\,d\alpha(t) = \int_a^b \left(\sum_{j=1}^{n} w_j f_j(t)\right) d\alpha(t).$$

From Schwarz's inequality,

$$\sum_{j=1}^{n} w_j f_j(t) \leq \underset{(4)}{\underline{\hspace{1cm}}} |\mathbf{f}(t)| \text{ for } t \in [a, b]. \tag{7.19}$$

Now $\sum_{j=1}^{n} w_j f_j(t)$ and $|\mathbf{w}|\,|\mathbf{f}(t)|$ are real-valued functions on $[a, b]$ that are in $\Re(\alpha)$. From (7.19) and Algebraic Property (c) of Riemann-Stieltjes integrals, it follows that

$$|\mathbf{w}|^2 = \int_a^b \left(\sum_{j=1}^{n} w_j f_j(t)\right) d\alpha(t) \leq \int_a^b |\mathbf{w}|\,|\mathbf{f}(t)|\,d\alpha(t) = |\mathbf{w}| \int_a^b |\mathbf{f}(t)|\,d\alpha(t).$$

Because $\mathbf{w} \neq 0$, $|\mathbf{w}|^2 \leq |\mathbf{w}| \int_a^b |\mathbf{f}(t)|\,d\alpha(t)$ implies that $|\mathbf{w}| \leq \int_a^b |\mathbf{f}(t)|\,d\alpha(t)$ which is equivalent to equation 7.17). ■

Acceptable responses are: (1) bounded, (2) Theorem 7.1.28, (3) $\in \Re(\alpha)$, (4) $|\mathbf{w}|$.

7.3.1 Rectifiable Curves

As an application of Riemann-Stieltjes integration on vector-valued functions we can prove a result that you assumed when you took elementary vector calculus. Recall the following definition.

Definition 7.3.7 *A continuous function γ from an interval $[a, b]$ into $\mathbb{R}^n$ is called a* ***curve*** *in $\mathbb{R}^n$ or a curve on $[a, b]$ in $\mathbb{R}^n$; if γ is one-to-one, then γ is called an* ***arc****, and if $\gamma(a) = (\beta)$, then γ is a* ***closed curve****.*

Remark 7.3.8 *In the definition of curve, we want to think of the curve as the actual mapping because the associated set of points in* $\mathbb{R}^n$ *is not uniquely determined by a particular mapping. As a simple example,* $\gamma_1(t) = (t,t)$ *and* $\gamma_2(t) = \left(t,t^2\right)$ *are two different mappings that give the same associated subset of* $\mathbb{R}^2$.

Given a curve γ on $[a,b]$, for any partition of $[a,b]$,

$$\mathcal{P} = \{x_0 = a, x_1, ..., x_{m-1}, x_m = b\}$$

let

$$\Lambda(\mathcal{P},\gamma) = \sum_{j=1}^{m} \left|\gamma\left(x_j\right) - \gamma\left(x_{j-1}\right)\right|.$$

Then $\Lambda(\mathcal{P},\gamma)$ is the length of a polygonal path having vertices $\gamma(x_0), \gamma(x_1), ..., \gamma(x_m)$ which, if conditions are right, gives an approximation for the length of the curve γ. For $\wp[a,b]$ the set of all partitions of $[a,b]$, it is reasonable to define the length of a curve γ as

$$\Lambda(\gamma) = \sup_{\mathcal{P}\in\wp[a,b]} \Lambda(\mathcal{P},\gamma);$$

if $\Lambda(\gamma) < \infty$, then γ is said to be **rectifiable**.

In various applications of mathematics integrating over curves becomes important. For this reason, we would like to have conditions under which we can determine when a given curve is rectifiable. We close this chapter with a theorem that tells us a condition under which Riemann integration can be used to determine the length of a rectifiable curve.

Theorem 7.3.9 *Suppose that* γ *is a curve on* $[a,b]$ *in* $\mathbb{R}^n$. *If* γ' *is continuous on* $[a,b]$, *then* γ *is rectifiable and*

$$\Lambda(\gamma) = \int_a^b \left|\gamma'(t)\right| dt.$$

Proof. Suppose that γ is a curve on $[a,b]$ in $\mathbb{R}^n$ such that γ' is continuous. From the Fundamental Theorem of Calculus and Theorem 7.1.39, for $\left[x_{j-1},x_j\right] \subset [a,b]$,

$$\left|\gamma\left(x_j\right) - \gamma\left(x_{j-1}\right)\right| = \left|\int_{x_{j-1}}^{x_j} \gamma'(t)\,dt\right| \le \int_{x_{j-1}}^{x_j} \left|\gamma'(t)\right| dt.$$

Hence, for any partition of $[a, b]$, $\mathcal{P} = \{x_0 = a, x_1, ..., x_{m-1}, x_m = b\}$,

$$\Lambda(\mathcal{P}, \gamma) \le \int_a^b |\gamma'(t)|\, dt$$

from which it follows that

$$\Lambda(\gamma) \le \int_a^b |\gamma'(t)|\, dt \tag{7.20}$$

Let $\varepsilon > 0$ be given. By the Uniform Continuity Theorem, γ' is uniformly continuous on $[a, b]$. Hence, there exists a $\delta > 0$ such that

$$|s - t| < \delta \Rightarrow |\gamma'(s) - \gamma'(t)| < \frac{\varepsilon}{2(b-a)}. \tag{7.21}$$

Choose $\mathcal{P} = \{x_0 = a, x_1, ..., x_{m-1}, x_m = b\} \in \wp[a, b]$ be such that mesh $\mathcal{P} < \delta$. It follows from (7.21) and the (other) triangular inequality that

$$t \in [x_{j-1}, x_j] \Rightarrow |\gamma'(t)| \le |\gamma'(x_j)| + \frac{\varepsilon}{2(b-a)}.$$

Thus,

$$\begin{aligned}
\textstyle\int_{x_{j-1}}^{x_j} |\gamma'(t)|\, dt &\le |\gamma'(x_j)|\, \Delta x_j + \frac{\varepsilon}{2(b-a)} \Delta x_j \\
&= \left| \textstyle\int_{x_{j-1}}^{x_j} [\gamma'(t) + \gamma(x_j) - \gamma'(t)]\, dt \right| + \frac{\varepsilon}{2(b-a)} \Delta x_j \\
&\le \left| \textstyle\int_{x_{j-1}}^{x_j} \gamma'(t)\, dt \right| + \left| \textstyle\int_{x_{j-1}}^{x_j} [\gamma(x_j) - \gamma'(t)]\, dt \right| + \frac{\varepsilon}{2(b-a)} \Delta x_j \\
&\le |\gamma(x_j) - \gamma(x_{j-1})| + 2\left(\frac{\varepsilon}{2(b-a)} \Delta x_j \right) \\
&= |\gamma(x_j) - \gamma(x_{j-1})| + \frac{\varepsilon}{(b-a)} \Delta x_j.
\end{aligned}$$

Summing the inequalities for $j = 1, 2, ...m$ yields that

$$\int_a^b |\gamma'(t)|\, dt \le \Lambda(\mathcal{P}, \gamma) + \varepsilon. \tag{7.22}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\int_a^b |\gamma'(t)|\, dt \le \Lambda(\mathcal{P}, \gamma)$.

Combining the inequalities (7.20) and (7.21) leads to the desired conclusion. ■

7.4 Problem Set G

1. Let $f(x) = x^2$, $g(x) = \lfloor 2x \rfloor$, and, for $n \in \mathbb{J}$, $\mathcal{P}_n$ denote the partition of $[0, 2]$ that subdivides the interval into n subintervals of equal length. Find each of the following.

   (a) $U(\mathcal{P}_3, f)$
   (b) $U(\mathcal{P}_5, g)$
   (c) $L(\mathcal{P}_4, f)$
   (d) $L(\mathcal{P}_6, g)$

2. For $f(x) = 2x^2 + 1$, $\alpha(t) = t + \lfloor 3t \rfloor$ and Δ the subdivision of $[0, 1]$ consisting of 4 subintervals of equal length, find $U(\Delta, f, \alpha)$ and $L(\Delta, f, \alpha)$.

3. For $f(x) = 3x$ in $\left[-\frac{1}{2}, 1\right]$, $\alpha(t) = t$, and $\mathcal{P} = \left\{-\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{2}, 1\right\}$, find $U(\mathcal{P}, f, \alpha)$ and $L(\mathcal{P}, f, \alpha)$.

4. Suppose that the function f in bounded on the interval $[a, b]$ and $g = kf$ for a fixed negative real number k. Prove that $\sup_{x \in I} g(x) = k \inf_{x \in I} f(x)$.

5. Suppose that the function f in bounded on the interval $I = [a, b]$ and $g = kf$ for a fixed negative integer k. Show that

$$\underline{\int_a^b} g(x)\,dx = k \overline{\int_a^b} f(x)\,dx.$$

6. Suppose that the functions f, f_1, and f_2 are bounded and defined on the closed interval $I = [a, b]$ and α is a function that is defined and monotonically increasing in I. Prove each of the following:

   (a) If $h = f_1 + f_2$, then $\underline{\int_a^b} h(x)\,d\alpha(x) \geq \underline{\int_a^b} f_1(x)\,d\alpha(x) + \underline{\int_a^b} f_2(x)\,d\alpha(x)$
   (b) If $h = f_1 + f_2$, then $\overline{\int_a^b} h(x)\,d\alpha(x) \leq \overline{\int_a^b} f_1(x)\,d\alpha(x) + \overline{\int_a^b} f_2(x)\,d\alpha(x)$
   (c) If $f_1(x) \leq f_2(x)$ for all $x \in I$, then $\underline{\int_a^b} f_1(x)\,d\alpha(x) \leq \underline{\int_a^b} f_2(x)\,d\alpha(x)$
   (d) If $f_1(x) \leq f_2(x)$ for all $x \in I$, then $\overline{\int_a^b} f_1(x)\,d\alpha(x) \leq \overline{\int_a^b} f_2(x)\,d\alpha(x)$.

(e) If $a < b < c$ and f is bounded on $I^* = \{x : a \leq x \leq c\}$ and α is monotonically increasing on I^*, then

$$\overline{\int_a^c} f(x)\, d\alpha(x) = \overline{\int_a^b} f(x)\, d\alpha(x) + \overline{\int_b^c} f(x)\, d\alpha(x).$$

7. Suppose that f is a bounded function on $I = [a, b]$. Let $M = \sup\limits_{x \in I} f(x)$, $m = \inf\limits_{x \in I} f(x)$, $M^* = \sup\limits_{x \in I} |f(x)|$, and $m^* = \inf\limits_{x \in I} |f(x)|$.

   (a) Show that $M^* - m^* \leq M - m$.

   (b) If f and g are nonnegative bounded functions on I, $N = \sup\limits_{x \in I} g(x)$, and $n = \inf\limits_{x \in I} g(x)$, show that

$$\sup_{x \in I} (fg)(x) - \inf_{x \in I} (fg)(x) \leq MN - mn.$$

8. Suppose that f is bounded and Riemann integrable on $I = [a, b]$.

   (a) Prove that $|f|$ is Riemann integrable on I.

   (b) Show that $\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx$.

9. Suppose that f and g are nonnegative, bounded and Riemann integrable on $I = [a, b]$. Prove that fg is Riemann integrable on I.

10. Let $A = \left\{ \frac{j}{2^n} : n, j \in \mathbb{J} \wedge j < 2^n \wedge 2 \nmid j \right\}$ and

$$f(x) = \begin{cases} \dfrac{1}{2^n} & \text{, if } \quad x \in A \\ 0 & \text{, if } \quad x \in [0,1] - A \end{cases}.$$

Is f Riemann integrable on $[0, 1]$. Carefully state and prove your conclusion.

11. Let $f(x) = x^2$ and $\alpha(t) = \lfloor 3t \rfloor$ where $\lfloor .. \rfloor$ denotes the greatest integer function.

(a) For the partition $\mathcal{P} = \left\{ x_j = \frac{2j}{3} : j = 0, 1, 2, 3 \right\}$ with associated subdivision $\Delta = \{I_1, I_2, I_3\}$, find $U(\mathcal{P}, f, \alpha)$.

(b) If $\mathcal{P}^* = \left\{ u_j = \frac{j}{3} : j = 0, 1, 2 \right\} \cup \left\{ u_{j+2} = \frac{2}{3} + \frac{2j}{9} : j = 1, 2, 3, 4, 5, 6 \right\}$ and Δ^* denotes the associated subdivision of $[0, 2]$, then $\mathcal{P}^*$ is a refinement of $\mathcal{P}$. For each $k \in \{1, 2, 3\}$, let $\Delta(k)$ be the subdivision of I_k consisting of the elements of Δ^* that are contained in I_k. Find $L(\Delta(2), f, \alpha)$.

12. For $a < b$, let $\mathcal{C}([a, b])$ denote the set of real-valued functions that are continuous on the interval $I = [a, b]$. For $f, g \in \mathcal{C}([a, b])$, set

$$d(f, g) \underset{def}{=} \int_a^b |f(x) - g(x)|\, dx.$$

Prove that $(\mathcal{C}([a, b]), d)$ is a metric space.

13. If f is monotonically increasing on an interval $I = [a, b]$, prove that f is Riemann integrable. Hint: Appeal to the Integrability Criterion.

14. For nonzero real constants $c_1, c_2, ..., c_n$, let $f(x) = \sum_{j=1}^{n} c_j \lfloor x \rfloor \chi_{[j, j+1)}(x)$, where $\lfloor \cdot \rfloor$ denotes the greatest integer function and χ denotes the characteristic function on $\mathbb{R}$. Is f Riemann integrable on $\mathbb{R}$? Carefully justify the position taken; if yes, find the value of the integral.

15. Prove that if a function f is “R” integrable (see Remark 7.1.27) on the interval $I = [a, b]$, then f is Riemann integrable on I.

16. Suppose that f and g are functions that are positive and continuous on an interval $I = [a, b]$. Prove that there is a number $\zeta \in I$ such that

$$\int_a^b f(x) g(x)\, dx = f(\zeta) \int_a^b g(x)\, dx.$$

17. For $a < b$, let $I = [a, b]$. If the function f is continuous on $I - \{c\}$, for a fixed $c \in (a, b)$, and bounded on I, prove that f is Riemann integrable on I.

18. Suppose that f in integrable on $I = [a, b]$ and
$$(\exists m)\,(\exists M)\,(m > 0 \wedge M > 0 \wedge (\forall x)\,(x \in I \Rightarrow m \leq f(x) \leq M))\,.$$
Prove $\dfrac{1}{f(x)} \in \Re$ on I.

19. For $f(x) = x^2 + 2x$, verify the Mean-Value Theorem for integrals in the interval $[1, 4]$.

20. Find $\displaystyle\int\limits_1^{\sin x^3} e^{\left(t^3+1\right)} dt$.

21. For $x > 0$, let $G(x) = \displaystyle\int_{\sqrt{x}}^{e^{x^2}} \sin^9 3t\ dt$. Make use of the First Fundamental Theorem of Calculus and the Chain Rule to find $G'(x)$. Show your work carefully.

22. Suppose that $f \in \Re$ and $g \in \Re$ on $I = [a, b]$. Then each of f^2, g^2, and fg are Riemann integrable on I. Prove the Cauchy-Schwarz inequality:
$$\left(\int_a^b f(x)\,g(x)\,dx\right)^2 \leq \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right).$$
[Note that, for $\alpha = \int_a^b f^2(x)\,dx$, $\beta = \int_a^b f(x)\,g(x)\,dx$, and $\gamma = \int_a^b g^2(x)\,dx$, $\alpha^2 x + 2\beta x + \gamma$ is nonnegative for all real numbers x.]

23. For $f(x) = \ln x = \int_1^x \dfrac{dt}{t}$ for $x > 0$ and a and b positive real numbers, prove each of the following.
    - (a) $\ln(ab) = \ln(a) + \ln(b)$,
    - (b) $\ln(a/b) = \ln(a) - \ln(b)$,
    - (c) $\ln(a^r) = r \cdot \ln(a)$ for every rational number r,
    - (d) $f'(x) = \dfrac{1}{x}$,
    - (e) f is increasing and continuous on $I = \{x : 0 < x < +\infty\}$, and
    - (f) $\dfrac{1}{2} \leq \ln(2) \leq 1$,

Chapter 8

Sequences and Series of Functions

Given a set A, a sequence of elements of A is a function $F : \mathbb{J} \to A$; rather than using the notation $F(n)$ for the elements that have been selected from A, since the domain is always the natural numbers, we use the notational convention $a_n = F(n)$ and denote sequences in any of the following forms:

$$\{a_n\}_{n=1}^{\infty}, \qquad \{a_n\}_{n\in\mathbb{J}}, \qquad \textit{or} \qquad a_1, a_2, a_3, a_4, \ldots..$$

Given any sequence $\{c_k\}_{k=1}^{\infty}$ of elements of a set A, we have an associated sequence of nth partial sums

$$\{s_n\}_{n=1}^{\infty} \text{ where } s_n = \sum_{k=1}^{n} c_k;$$

the symbol $\sum_{k=1}^{\infty} c_k$ is called a series (or infinite series). Because the function $g(x) = x - 1$ is a one-to-one correspondence from $\mathbb{J}$ into $\mathbb{J} \cup \{0\}$, i.e., $g : \mathbb{J} \overset{1-1}{\twoheadrightarrow} \mathbb{J} \cup \{0\}$, a sequence could have been defined as a function on $\mathbb{J} \cup \{0\}$. In our discussion of series, the symbolic descriptions of the sequences of nth partial sums usually will be generated from a sequence for which the first subscript is 0. The notation always makes the indexing clear, when such specificity is needed.

Thus far, our discussion has focused on sequences and series of complex (and real) numbers; i.e., we have taken $A = \mathbb{C}$ and $A = \mathbb{R}$. In this chapter, we take A to be the set of complex (and real) functions on $\mathbb{C}$ (and $\mathbb{R}$).

8.1 Pointwise and Uniform Convergence

The first important thing to note is that we will have different types of convergence to consider because we have "more variables." The first relates back to numerical sequences and series. We start with an example for which the work was done in Chapter 4.

Example 8.1.1 *For each* $n \in \mathbb{J}$*, let* $f_n(z) = z^n$ *where* $z \in \mathbb{C}$*. We can use results obtained earlier to draw some conclusions about the convergence of* $\{f_n(z)\}_{n=1}^{\infty}$*. In Lemma 4.4.2, we showed that,* ***for any fixed complex number*** z_0 *such that* $|z_0| < 1$*,* $\lim_{n\to\infty} z_0^n = 0$*. In particular, we showed that for* z_0*,* $0 < |z_0| < 1$*, if* $\varepsilon > 0$*, then taking*

$$M = M(\varepsilon, z_0) = \begin{cases} 1 & , \text{ for } \varepsilon \geq 1 \\ \left\lceil \dfrac{\ln(\varepsilon)}{\ln|z_0|} \right\rceil & , \text{ for } \varepsilon < 1 \end{cases}.$$

yields that $\left|z_0^n - 0\right| < \varepsilon$ *for all* $n > M$*. When* $z_0 = 0$*, we have the constant sequence. In offering this version of the statement of what we showed, I made a "not so subtle" change in format; namely, I wrote the former* $M(\varepsilon)$ *and* $M(\varepsilon, z_0)$*. The change was to stress that our discussion was tied to the fixed point. In terms of our sequence* $\{f_n(z)\}_{n=1}^{\infty}$*, we can say that for each fixed point* $z_0 \in \Omega = \{z \in \mathbb{C} : |z| < 1\}$*,* $\{f_n(z_0)\}_{n=1}^{\infty}$ *is convergent to* 0*. This gets us to some new terminology: For this example, if* $f(z) = 0$ *for all* $z \in \mathbb{C}$*, then we say that* $\{f_n(z_0)\}_{n=1}^{\infty}$ *is* ***pointwise convergent to*** f ***on*** Ω*.*

It is very important to keep in mind that our argument for convergence at each fixed point made clear and definite use of the fact that we had a point for which a known modulus was used in finding an $M(\varepsilon, z_0)$. It is natural to ask if the pointwise dependence was necessary. We will see that the answer depends on the nature of the sequence. For the sequence given in Example 8.1.1, the best that we will be able to claim over the set Ω is pointwise convergence. The associated sequence of nth partial sums for the functions in the previous example give us an example of a sequence of functions for which the pointwise limit is not a constant.

Example 8.1.2 *For* $a \neq 0$ *and each* $k \in \mathbb{J} \cup \{0\}$*, let* $f_k(z) = az^k$ *where* $z \in \mathbb{C}$*. In Chapter 4, our proof of the Convergence Properties of the Geometric Series*

Theorem showed that the associated sequence of n*th partial sums* $\{s_n(z)\}_{n=0}^{\infty}$ *was given by*

$$s_n(z) = \sum_{k=0}^{n} f_k(z) = \sum_{k=0}^{n} az^k = \frac{a\left(1 - z^{n+1}\right)}{1 - z}.$$

In view of Example 8.1.1, we see that for each fixed $z_0 \in \Omega = \{z \in \mathbb{C} : |z| < 1\}$, $\{s_n(z_0)\}_{n=1}^{\infty}$ *is convergent to* $\dfrac{a}{1 - z_0}$*. Thus,* $\{s_n(z)\}_{n=0}^{\infty}$ *is* ***pointwise convergent on*** $\boldsymbol{\Omega}$*. In terminology that is soon to be introduced, we more commonly say that "the series* $\sum_{k=0}^{\infty} az^k$ *is pointwise convergent on* Ω*."*

Our long term goal is to have an alternative way of looking at functions. In particular, we want a view that would give promise of transmission of nice properties, like continuity and differentiability. The following examples show that pointwise convergence proves to be insufficient.

Example 8.1.3 *For each* $n \in \mathbb{J}$*, let* $f_n(z) = \dfrac{n^2 z}{1 + n^2 z}$ *where* $z \in \mathbb{C}$*. For each fixed* z *we can use our properties of limits to find the pointwise limit of the sequences of functions. If* $z = 0$*, then* $\{f_n(0)\}_{n=1}^{\infty}$ *converges to* 0 *as a constant sequence of zeroes. If* z *is a fixed nonzero complex number, then*

$$\lim_{n\to\infty} \frac{n^2 z}{1 + n^2 z} = \lim_{n\to\infty} \frac{z}{\dfrac{1}{n^2} + z} = \frac{z}{z} = 1.$$

Therefore, $f_n \longrightarrow f$ *where* $f(z) = \begin{cases} 1 & , \text{for} \quad z \in \mathbb{C} - \{0\} \\ \\ 0 & , \text{for} \quad z = 0 \end{cases}$.

Remark 8.1.4 *From Theorem 4.4.3(c) or Theorem 3.20(d) of our text, we know that* $p > 0$ *and* $\alpha \in \mathbb{R}$*, implies that* $\displaystyle\lim_{n\to\infty} \frac{n^{\alpha}}{(1 + p)^n} = 0$*. Letting* $\zeta = \dfrac{1}{1 + p}$ *for* $p > 0$ *leads to the observation that*

$$\lim_{n\to\infty} n^{\alpha} \zeta^n = 0 \tag{8.1}$$

whenever $0 \leq \zeta < 1$ *and for any* $\alpha \in R$*. This is the form of the statement that is used by the author of our text in Example 7.6 where a sequence of functions for which the integral of the pointwise limit differs from the limit of the integrals of the functions in the sequence is given.*

Example 8.1.5 (7.6 in our text) *Consider the sequence* $\{f_n\}_{n=1}^{\infty}$ *of real-valued functions on the interval* $I = [0,1]$ *that is given by* $f_n(x) = nx\left(1-x^2\right)^n$ *for* $n \in \mathbb{J}$. *For fixed* $x \in I - \{0\}$, *taking* $\alpha = 1$ *and* $\zeta = \left(1-x^2\right)$ *in (8.1) yields that* $n\left(1-x^2\right)^n \longrightarrow 0$ *as* $n \to \infty$. *Hence,* $f_n \underset{I-\{0\}}{\longrightarrow} 0$. *Because* $f_n(0) = 0$ *for all* $n \in \mathbb{J}$, *we see that for each* $x \in I$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx\left(1-x^2\right)^n = 0.$$

In contrast to having the Riemann integral of the limit function over I *being* 0, *we have that*

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{n}{2n+2} = \frac{1}{2}.$$

Note that, since α *in Equation (8.1) can be any real number, the sequence of real functions* $g_n(x) = n^2x\left(1-x^2\right)^n$ *for* $n \in \mathbb{J}$ *converges pointwise to* 0 *on* I *with*

$$\int_0^1 g_n(x)\,dx = \frac{n^2}{2n+2} \to \infty \text{ as } n \to \infty.$$

This motivates the search for a stronger sense of convergence; namely, uniform convergence of a sequence (and, in turn, of a series) of functions. Remember that our application of the term "uniform" to continuity required much nicer behavior of the function than continuity at points. We will make the analogous shift in going from pointwise convergence to uniform convergence.

Definition 8.1.6 *A sequence of complex functions* $\{f_n\}_{n=1}^{\infty}$ ***converges pointwise to a function*** f *on a subset* Ω *of* $\mathbb{C}$, *written* $f_n \longrightarrow f$ *or* $f_n \underset{z\in\Omega}{\longrightarrow} f$, *if and only if the sequence* $\{f_n(z_0)\}_{n=1}^{\infty} \longrightarrow f(z_0)$ *for each* $z_0 \in \Omega$; *i.e., for each* $z_0 \in \Omega$

$$(\forall\varepsilon > 0)\,(\exists M = M(\varepsilon, z_0) \in \mathbb{J})\,(n > M(\epsilon, z_0) \Rightarrow |f_n(z_0) - f(z_0)| < \varepsilon).$$

Definition 8.1.7 *A sequence of complex functions* $\{f_n\}$ ***converges uniformly*** *to* f *on a subset* Ω *of* $\mathbb{C}$, *written* $f_n \rightrightarrows f$, *if and only if*

$$(\forall\varepsilon > 0)\,(\exists M = M(\varepsilon))\,(M \in \mathbb{J} \wedge (\forall n)\,(\forall z)\,(n > M \wedge z \in \Omega \Rightarrow |f_n(z) - f(z)| < \varepsilon)).$$

Remark 8.1.8 *Uniform convergence implies pointwise convergence. Given a sequence of functions, the only candidate for the uniform limit is the pointwise limit.*

Example 8.1.9 *The sequence considered in Example 8.1.1 exhibits the stronger sense of convergence if we restrict ourselves to compact subsets of* $\Omega = \{z \in \mathbb{C} : |z| < 1\}$*. For each* $n \in \mathbb{J}$*, let* $f_n(z) = z^n$ *where* $z \in \mathbb{C}$*. Then* $\{f_n(z)\}_{n=1}^{\infty}$ *is uniformly convergent to the constant function* $f(z) = 0$ *on any compact subset of* Ω*.*

Suppose $K \subset \Omega$ *is compact. From the Heine-Borel Theorem, we know that* K *is closed and bounded. Hence, there exists a positive real number* r *such that* $r < 1$ *and* $(\forall z)(z \in K \Rightarrow |z| \le r)$*. Let* $\Omega_r = \{z \in \mathbb{C} : |z| \le r\}$*. For* $\varepsilon > 0$*, let*

$$M = M(\varepsilon) = \begin{cases} 1 & , \text{ for } \varepsilon \ge 1 \\ \left\lceil \dfrac{\ln(\varepsilon)}{\ln r} \right\rceil & , \text{ for } \varepsilon < 1 \end{cases} .$$

Then $n > M \Rightarrow n > \dfrac{\ln(\varepsilon)}{\ln r} \Rightarrow n \ln r < \ln(\varepsilon)$ *because* $0 < r < 1$*. Consequently,* $r^n < \varepsilon$ *and it follows that*

$$|f_n(z) - 0| = \left|z^n\right| = |z|^n \le r^n < \varepsilon.$$

Since $\varepsilon > 0$ *was arbitrary, we conclude that* $f_n \underset{\Omega_r}{\rightrightarrows} f$*. Because* $K \subset \Omega_r$*,* $f_n \underset{K}{\rightrightarrows} f$ *as claimed.*

Excursion 8.1.10 *When we restrict ourselves to consideration of uniformly convergent sequences of real-valued functions on* $\mathbb{R}$*, the definition links up nicely to a graphical representation. Namely, suppose that* $f_n \underset{[a,b]}{\rightrightarrows} f$*. Then corresponding to any* $\varepsilon > 0$*, there exists a positive integer* M *such that* $n > M \Rightarrow |f_n(x) - f(x)| < \varepsilon$ *for all* $x \in [a, b]$*. Because we have real-valued functions on the interval, the inequality translates to*

$$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \text{ for all } x \in [a, b]. \tag{8.2}$$

Label the following figure to illustrate what is described in (8.2) and illustrate the implication for any of the functions f_n *when* $n > M$*.*

Remark 8.1.11 *The negation of the definition offers us one way to prove that a sequence of functions is not uniformly continuous. Given a sequence of functions* $\{f_n\}$ *that are defined on a subset* Ω *of* $\mathbb{C}$*, the convergence of* $\{f_n\}$ *to a function* f *on* Ω *is not uniform if and only if*

$$(\exists \varepsilon > 0)\,(\forall M)\,[M \in \mathbb{J} \Rightarrow \\ (\exists n)\left(\exists z_{M_n}\right)\left(n > M \wedge z_{M_n} \in \Omega \wedge \left| f_n\left(z_{M_n}\right) - f\left(z_{M_n}\right)\right| \geq \varepsilon\right)].$$

Example 8.1.12 *Use the definition to show that the sequence* $\left\{\frac{1}{nz}\right\}_{n=1}^{\infty}$ *is pointwise convergent, but not uniformly convergent, to the function* $f(x) = 0$ *on* $\Omega = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Suppose that z_0 *is a fixed element of* Ω*. For* $\varepsilon > 0$*, let* $M = M(\varepsilon, z_0) = \left\lceil \frac{1}{|z_0|\,\varepsilon} \right\rceil$*. Then* $n > M \Rightarrow n > \frac{1}{|z_0|\,\varepsilon} \Rightarrow \frac{1}{n\,|z_0|} < \varepsilon$ *because* $|z_0| > 0$*. Hence,*

$$\left|\frac{1}{nz_0} - 0\right| = \frac{1}{n\,|z_0|} < \varepsilon.$$

Since $\varepsilon > 0$ *was arbitrary, we conclude that* $\left\{\frac{1}{nz_0}\right\}_{n=1}^{\infty}$ *is convergent to* 0 *for each* $z_0 \in \Omega$*. Therefore,* $\left\{\frac{1}{nz}\right\}_{n=1}^{\infty}$ *is pointwise convergent on* Ω.

On the other hand, let $\varepsilon = \frac{1}{2}$ *and for each* $n \in \mathbb{J}$*, set* $z_n = \frac{1}{n+1}$*. Then*

$z_n \in \Omega$ and

$$\left|\frac{1}{nz_n}-0\right| = \left|\frac{1}{n\left(\dfrac{1}{n+1}\right)}\right| = 1+\frac{1}{n} \geq \varepsilon.$$

Hence, $\left\{\dfrac{1}{nz}\right\}_{n=1}^{\infty}$ is not uniformly convergent on Ω.

Example 8.1.13 *Prove that the sequence $\left\{\dfrac{1}{1+nz}\right\}_{n=1}^{\infty}$ converges uniformly for $|z| \geq 2$ and does not converge uniformly in $\Omega^* = \{z \in \mathbb{C} : |z| \leq 2\} - \left\{-\dfrac{1}{n} : n \in \mathbb{J}\right\}$.*

Let $\Omega = \{z \in \mathbb{C} : |z| \geq 2\}$ and, for each $n \in \mathbb{J}$, let $f_n(z) = \dfrac{1}{1+nz}$. From the limit properties of sequences, $\{f_n(z)\}_{n=1}^{\infty}$ is pointwise convergent on $\mathbb{C}$ to

$$f(z) = \begin{cases} 0 & , for \quad z \in \mathbb{C}-\{0\} \\ 1 & , for \quad z = 0 \end{cases}.$$

Thus, the pointwise limit of $\{f_n(z)\}_{n=1}^{\infty}$ on Ω is the constant function 0. For $\varepsilon > 0$, let $M = M(\varepsilon) = \left\lceil \dfrac{1}{2}\left(\dfrac{1}{\varepsilon}+1\right)\right\rceil$. Then $n > M \Rightarrow n > \dfrac{1}{2}\left(\dfrac{1}{\varepsilon}+1\right) \Rightarrow \dfrac{1}{2n-1} < \varepsilon$ because $n > 1$. Furthermore, $|z| \geq 2 \Rightarrow n|z| \geq 2n \Rightarrow n|z|-1 \geq 2n-1 > 0$. Hence, $|z| \geq 2 \wedge n > M \Rightarrow$

$$|f_n(z)-0| = \left|\frac{1}{1+nz}\right| \leq \frac{1}{|n|z|-1|} \leq \frac{1}{n|z|-1} \leq \frac{1}{2n-1} < \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, we conclude that $f_n \underset{\Omega}{\rightrightarrows} 0$.

On the other hand, let $\varepsilon = \dfrac{1}{2}$ and, corresponding to each $n \in \mathbb{J}$, set $z_n = \dfrac{1}{n}$. Then $z_n \in \Omega^*$ and

$$|f_n(z_n)-0| = \left|\frac{1}{1+n\left(\dfrac{1}{n}\right)}\right| = \frac{1}{2} \geq \varepsilon.$$

Hence, $\{f_n(z)\}_{n=1}^{\infty}$ *is not uniformly convergent in* Ω^*.

Excursion 8.1.14 *Use the definition to prove that the sequence* $\{z^n\}$ *is not uniformly convergent in* $|z| < 1$.

Hopefully, you thought to make use of the choices $\zeta_n = \left(1 - \dfrac{1}{n}\right)$ that could be related back to e^{-1}.

Using the definition to show that a sequence of functions is not uniformly convergent, usually, involves exploitation of "bad points." For Examples 8.1.12 and 8.1.13, the exploitable point was $x = 0$ while, for Example 8.1.14, it was $x = 1$.

Because a series of functions is realized as the sequence of nth partial sums of a sequence of functions, the definitions of pointwise and uniform convergence of series simply make statements concerning the nth partial sums. On the other hand, we add the notion of absolute convergence to our list.

Definition 8.1.15 *Corresponding to the sequence* $\{c_k(z)\}_{k=0}^{\infty}$ *of complex-valued functions on a set* $\Omega \subset \mathbb{C}$, *let*

$$S_n(z) = \sum_{k=0}^{n} c_k(z)$$

denote the sequence of nth *partial sums. Then*

(a) *the series* $\sum_{k=0}^{\infty} c_k(z)$ *is* ***pointwise convergent on*** Ω ***to*** S *if and only if, for each* $z_0 \in \Omega$, $\{S_n(z_0)\}_{n=0}^{\infty}$ *converges to* $S(z_0)$*; and*

(b) *the series* $\sum_{k=0}^{\infty} c_k(z)$ *is* ***uniformly convergent on*** Ω ***to*** S *if and only if* $S_n \underset{\Omega}{\rightrightarrows} S$.

Definition 8.1.16 *Corresponding to the sequence* $\{c_k(z)\}_{k=0}^{\infty}$ *of complex-valued functions on a set* $\Omega \subset \mathbb{C}$*, the series* $\sum_{k=0}^{\infty} c_k(z)$ *is* ***absolutely convergent on*** $\boldsymbol{\Omega}$ *if and only if* $\sum_{k=0}^{\infty} |c_k(z)|$ *is convergent for each* $z \in \Omega$*.*

Excursion 8.1.17 *For* $a \neq 0$ *and* $k \in \mathbb{J} \cup \{0\}$*, let* $c_k(z) = az^k$*. In Example 8.1.2, we saw that*

$$\sum_{k=0}^{\infty} c_k(z) = \sum_{k=0}^{\infty} az^k$$

is pointwise convergent for each $z_0 \in \Omega = \{z \in \mathbb{C} : |z| < 1\}$ *to* $a(1 - z_0)^{-1}$*. Show that*

(i) $\sum_{k=0}^{\infty} c_k(z)$ *is absolutely convergent for each* $z_0 \in \Omega$*;*

(ii) $\sum_{k=0}^{\infty} c_k(z)$ *is uniformly convergent on every compact subset* K *of* Ω*;*

(iii) $\sum_{k=0}^{\infty} c_k(z)$ *is not uniformly convergent on* Ω*.*

For part (i), hopefully you noticed that the formula derived for the proof of the Convergence Properties of the Geometric Series applied to the real series that results from replacing az^k with $|a|\,|z|^k$. Since $\sum_{k=0}^{n} |a|\,|z|^k = \dfrac{|a|\left(1 - |z|^{n+1}\right)}{1 - |z|}$, we conclude that $\left\{\sum_{k=0}^{n}|a|\,|z|^{k}\right\}_{n=0}^{\infty} \longrightarrow \dfrac{|a|}{1-|z|}$ for each $z \in C$ such that $|z| < 1$; i.e., $\sum_{k=0}^{\infty} c_k(z) = \sum_{k=0}^{\infty} az^k$ is absolutely convergent for each $z \in \Omega$. To show part (ii) it is helpful to make use of the fact that if K is a compact subset of Ω then there exists a positive real number r such that $r < 1$ and $K \subset \Omega_r = \{z \in \mathbb{C} : |z| \le r\}$. The uniform convergence of $\sum_{k=0}^{\infty} c_k(z)$ on Ω_r then yields uniform convergence on K. For $S_n(z) = \sum_{k=0}^{n} c_k(z) = \sum_{k=0}^{n} az^k = \dfrac{a\left(1-z^{n+1}\right)}{1-z}$ and $S(z) = \dfrac{a}{1-z}$, you should have noted that $|S_n(z) - S(z)| \le \dfrac{|a|\,r^{n+1}}{1-r}$ for all $z \in \Omega_r$ which leads to $M(\varepsilon) = \max\left\{1, \left\lceil \dfrac{\ln\left(\varepsilon(1-r)\,|a|^{-1}\right)}{\ln r} - 1 \right\rceil\right\}$ as one possibility for justifying the uniform convergence. Finally, with (iii), corresponding to each $n \in \mathbb{J}$, let $z_n = \left(1 - \dfrac{1}{n+1}\right)$; then $z_n \in \Omega$ for each n and $S_n(z_n) - S(z_n) = (n+1)\,|a|\left(1 - \dfrac{1}{n+1}\right)^{n+1}$ can be used to justify that we do not have uniform convergence.

8.1.1 Sequences of Complex-Valued Functions on Metric Spaces

In much of our discussion thus far and in numerous results to follow, it should become apparent that the properties claimed are dependent on the properties of the codomain for the sequence of functions. Indeed our original statement of the definitions of pointwise and uniform convergence require bounded the distance between images of points from the domain while not requiring any "nice behavior relating the points of the domain to each other." To help you keep this in mind, we state the definitions again for sequences of functions on an arbitrary metric space.

Definition 8.1.18 *A sequence of complex functions* $\{f_n\}_{n=1}^{\infty}$ ***converges pointwise to a function*** f *on a subset* Ω *of a metric space* (X, d)*, written* $f_n \longrightarrow f$ *or* $f_n \underset{w\in\Omega}{\longrightarrow} f$*, if and only if the sequence* $\{f_n(w_0)\}_{n=1}^{\infty} \longrightarrow f(w_0)$ *for each* $w_0 \in \Omega$*; i.e., for each* $w_0 \in \Omega$

$$(\forall \varepsilon > 0)\,(\exists M = M(\varepsilon, w_0) \in \mathbb{J})\,(n > M(\epsilon, w_0) \Rightarrow |f_n(w_0) - f(w_0)| < \varepsilon).$$

Definition 8.1.19 *A sequence of complex functions* $\{f_n\}$ ***converges uniformly*** *to* f *on a subset* Ω *of a metric space* (X, d)*, written* $f_n \rightrightarrows f$ *on* Ω *or* $f_n \underset{\Omega}{\rightrightarrows} f$*, if and only if*

$$(\forall \varepsilon > 0)\,(\exists M = M(\varepsilon))\,[M \in \mathbb{J} \wedge (\forall n)\,(\forall w)\,(n > M \wedge w \in \Omega \\ \Rightarrow |f_n(w) - f(w)| < \varepsilon)].$$

8.2 Conditions for Uniform Convergence

We would like some other criteria that can allow us to make decisions concerning the uniform convergence of given sequences and series of functions. In addition, if can be helpful to have a condition for uniform convergence that does not require knowledge of the limit function.

Definition 8.2.1 *A sequence* $\{f_n\}_{n=1}^{\infty}$ *of complex-valued functions satisfies the* ***Cauchy Criterion for Convergence*** *on* $\Omega \subset \mathbb{C}$ *if and only if*

$$(\forall \varepsilon > 0)\,(\exists M \in \mathbb{J})\,[(\forall n)\,(\forall m)\,(\forall z)\,(n > M \wedge m > M \wedge z \in \Omega \\ \Rightarrow |f_n(z) - f_m(z)| < \varepsilon)].$$

Remark 8.2.2 *Alternatively, when a sequence satisfies the Cauchy Criterion for Convergence on a subset* $\Omega \subset \mathbb{C}$ *it may be described as being* ***uniformly Cauchy on*** Ω *or simply as being Cauchy.*

In Chapter 4, we saw that in $\mathbb{R}^n$ being convergent was equivalent to being Cauchy convergent. The same relationship carries over to uniform convergence of functions.

Theorem 8.2.3 *Let* $\{f_n\}_{n=1}^{\infty}$ *denote a sequence of complex-valued functions on a set* $\Omega \subset \mathbb{C}$*. Then* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly on* Ω *if and only if* $\{f_n\}_{n=1}^{\infty}$ *satisfies the Cauchy Criterion for Convergence on* Ω*.*

Space for scratch work.

Proof. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of complex-valued functions on a set $\Omega \subset \mathbb{C}$ that converges uniformly on Ω to the function f and let $\varepsilon > 0$ be given. Then there exists $M \in \mathbb{J}$ such that $n > M$ implies that $|f_n(z) - f(z)| < \frac{\varepsilon}{2}$ for all $z \in \Omega$. Taking any other $m > M$ also yields that $|f_m(z) - f(z)| < \frac{\varepsilon}{2}$ for all $z \in \Omega$. Hence, for $m > M \wedge n > M$,

$$\begin{aligned} |f_m(z) - f_n(z)| &= |(f_m(z) - f(z)) - (f_n(z) - f(z))| \\ &\leq |f_m(z) - f(z)| + |f_n(z) - f(z)| < \varepsilon \end{aligned}$$

for all $z \in \Omega$. Therefore, $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy on Ω.

Suppose the sequence $\{f_n\}_{n=1}^{\infty}$ of complex-valued functions on a set $\Omega \subset \mathbb{C}$ satisfies the Cauchy Criterion for Convergence on Ω and let $\varepsilon > 0$ be given. For $z \in \Omega$, $\{f_n(z)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$; because $\mathbb{C}$ is complete, it follows that $\{f_n(z)\}_{n=1}^{\infty}$ is convergent to some $\zeta_z \in \mathbb{C}$. Since $z \in \Omega$ was arbitrary, we can define a function $f : \Omega \longrightarrow \mathbb{C}$ by $(\forall z)(z \in \Omega \Rightarrow f(z) = \zeta_z)$. Then, f is the pointwise limit of $\{f_n\}_{n=1}^{\infty}$. Because $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy, there exists an $M \in \mathbb{J}$ such that $m > M$ and $n > M$ implies that

$$|f_n(z) - f_m(z)| < \frac{\varepsilon}{2} \text{ for all } z \in \Omega.$$

Suppose that $n > M$ is fixed and $z \in \Omega$. Since $\lim_{m \to \infty} f_m(\xi) = f(\xi)$ for each $\xi \in \Omega$, there exists a positive integer $M^* > M$ such that $m > M^*$ implies that $|f_m(z) - f(z)| < \frac{\varepsilon}{2}$. In particular, we have that $|f_{M^*+1}(z) - f(z)| < \frac{\varepsilon}{2}$. Therefore,

$$\begin{aligned} |f_n(z) - f(z)| &= |f_n(z) - f_{M^*+1}(z) + f_{M^*+1}(z) - f(z)| \\ &\leq |f_n(z) - f_{M^*+1}(z)| + |f_{M^*+1}(z) - f(z)| < \varepsilon. \end{aligned}$$

But $n > M$ and $z \in \Omega$ were both arbitrary. Consequently,

$$(\forall n)\,(\forall z)\,(n > M \wedge z \in \Omega \Rightarrow |\,f_n\,(z) - f\,(z)| < \varepsilon)\,.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $f_n \underset{\Omega}{\rightrightarrows} f$. ■

Remark 8.2.4 *Note that in the proof just given, the positive integer* M^* *was dependent on the point* z *and the* ε*; i.e.,* $M^* = M^*\,(\varepsilon, z)$*. However, the final inequality obtained via the intermediate travel through information from* M^**,* $|\,f_n\,(z) - f\,(z)| < \varepsilon$*, was independent of the point* z*. What was illustrated in the proof was a process that could be used repeatedly for each* $z \in \Omega$*.*

Remark 8.2.5 *In the proof of both parts of Theorem 8.2.3, our conclusions relied on properties of the codomain for the sequence of functions. Namely, we used the metric on* $\mathbb{C}$ *and the fact that* $\mathbb{C}$ *was complete. Consequently, we could allow* Ω *to be any metric space and claim the same conclusion. The following corollary formalizes that claim.*

Corollary 8.2.6 *Let* $\{\,f_n\}_{n=1}^{\infty}$ *denote a sequence of complex-valued functions defined on a subset* Ω *of a metric space* (X, d)*. Then* $\{\,f_n\}_{n=1}^{\infty}$ *converges uniformly on* Ω *if and only if* $\{\,f_n\}_{n=1}^{\infty}$ *satisfies the Cauchy Criterion for Convergence on* Ω*.*

Theorem 8.2.7 *Let* $\{\,f_n\}_{n=1}^{\infty}$ *denote a sequence of complex-valued functions on a set* $\Omega \subset \mathbb{C}$ *that is pointwise convergent on* Ω *to the function* f*; i.e.,*

$$\lim_{n\to\infty} f_n\,(z) = f\,(z)\,;$$

and, for each $n \in \mathbb{J}$*, let* $M_n = \sup_{z\in\Omega} |\,f_n\,(z) - f\,(z)|$*. Then* $f_n \underset{\Omega}{\rightrightarrows} f$ *if and only if* $\lim_{n\to\infty} M_n = 0$*.*

Use this space to fill in a proof for Theorem 8.2.7.

Theorem 8.2.8 (Weierstrass M-Test) *For each $n \in \mathbb{J}$, let $u_n(w)$ be a complex-valued function that is defined on a subset Ω of a metric space (X, d). Suppose that there exists a sequence of real constants $\{M_n\}_{n=1}^{\infty}$ such that $|u_n(w)| \leq M_n$ for all $w \in \Omega$ and for each $n \in \mathbb{J}$. If the series $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} u_n(w)$ and $\sum_{n=1}^{\infty} |u_n(w)|$ converge uniformly on Ω.*

Excursion 8.2.9 *Fill in what is missing in order to complete the following proof of the Weierstrass M-Test.*

Proof. Suppose that $\{u_n(w)\}_{n=1}^{\infty}$, Ω, and $\{M_n\}_{n=1}^{\infty}$ are as described in the hypotheses. For each $n \in J$, let

$$S_n(w) = \sum_{k=1}^{n} u_k(w) \text{ and } T_n(w) = \sum_{k=1}^{n} |u_k(w)|$$

and suppose that $\varepsilon > 0$ is given. Since $\sum_{n=1}^{\infty} M_n$ converges and $\{M_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $\left\{\sum_{k=1}^{n} M_k\right\}_{n=1}^{\infty}$ is a convergent sequence of real numbers. In view of the completeness of the reals, we have that $\left\{\sum_{k=1}^{n} M_k\right\}_{n=1}^{\infty}$ is $\underset{(1)}{\underline{\hspace{3cm}}}$. Hence, there exists a positive integer K such that $n > K$ implies that

$$\sum_{k=n+1}^{n+p} M_k < \varepsilon \text{ for each } p \in \mathbb{J}.$$

Since $|u_k(w)| \leq M_k$ for all $w \in \Omega$ and for each $k \in \mathbb{J}$, we have that

$$\left|T_{n+p}(w) - T_n(w)\right| = \left|\sum_{k=n+1}^{n+p} |u_k(w)|\right| = \sum_{k=n+1}^{n+p} |u_k(w)| \text{ for all } w \in \Omega.$$

Therefore, $\{T_n\}_{n=1}^{\infty}$ is $\underset{(2)}{\underline{\hspace{3cm}}}$ in Ω. It follows from the $\underset{(3)}{\underline{\hspace{4.5cm}}}$ that

$$\underset{(4)}{\underline{\hspace{6cm}}} = \underset{(5)}{\underline{\hspace{3cm}}} \leq \sum_{k=n+1}^{n+p} |u_k(w)| \leq \sum_{k=n+1}^{n+p} M_k < \varepsilon$$

for all $w \in \Omega$. Hence, $\{S_n\}_{n=1}^{\infty}$ is uniformly Cauchy in Ω. From Corollary 8.2.6, we conclude that $\underset{(6)}{\underline{\hspace{6cm}}}$. ■

Acceptable responses include: (1) Cauchy, (2) uniformly Cauchy, (3) triangular inequality, (4) $\left|S_{n+p}(w) - S_n(w)\right|$, (5) $\left|\sum_{k=n+1}^{n+p} u_k(w)\right|$, and (6) $\sum_{n=1}^{\infty} u_n(w)$ and $\sum_{n=1}^{\infty} |u_n(w)|$ converge uniformly on Ω.

Excursion 8.2.10 *Construct an example to show that the converse of the Weierstrass M-Test need not hold.*

8.3 Property Transmission and Uniform Convergence

We have already seen that pointwise convergence was not sufficient to transmit the property of continuity of each function in a sequence to the limit function. In this section, we will see that uniform convergence overcomes that drawback and allows for the transmission of other properties.

Theorem 8.3.1 *Let $\{f_n\}_{n=1}^{\infty}$ denote a sequence of complex-valued functions defined on a subset Ω of a metric space (X, d) such that $f_n \underset{\Omega}{\rightrightarrows} f$. For w a limit point of Ω and each $n \in \mathbb{J}$, suppose that*

$$\lim_{\substack{t \to w \\ t \in \Omega}} f_n(t) = A_n.$$

Then $\{A_n\}_{n=1}^{\infty}$ converges and $\lim_{t \to w} f(t) = \lim_{n \to \infty} A_n$.

Excursion 8.3.2 *Fill in what is missing in order to complete the following proof of the Theorem.*

Proof. *Suppose that the sequence $\{f_n\}_{n=1}^{\infty}$ of complex-valued functions defined on a subset Ω of a metric space (X, d) is such that $f_n \underset{\Omega}{\rightrightarrows} f$, w is a limit point of Ω and, for each $n \in \mathbb{J}$, $\lim_{t \to w} f_n(t) = A_n$. Let $\varepsilon > 0$ be given. Since $f_n \underset{\Omega}{\rightrightarrows} f$,* by Corollary 8.2.6, $\{f_n\}_{n=1}^{\infty}$ is $\underset{(1)}{\underline{\hspace{3cm}}}$ on Ω. Hence, there exists a positive integer M such that $\underset{(2)}{\underline{\hspace{3cm}}}$ implies that

$$|f_n(t) - f_m(t)| < \frac{\varepsilon}{3} \text{ for all } \underset{(3)}{\underline{\hspace{3cm}}}.$$

Fix m and n such that $m > M$ and $n > M$. Since $\lim_{t \to w} f_k(t) = A_k$ for each $k \in \mathbb{J}$, it follows that there exists a $\delta > 0$ such that $0 < d(t, w) < \delta$ implies that

$$|f_m(t) - A_m| < \frac{\varepsilon}{3} \text{ and } \underset{(4)}{\underline{\hspace{6cm}}}$$

From the triangular inequality,

$$|A_n - A_m| \leq |A_n - f_n(t)| + \left|\underset{(5)}{\underline{\hspace{3cm}}}\right| + |f_m(t) - A_m| < \varepsilon.$$

Since m and n were arbitrary, for each $\varepsilon > 0$ there exists a positive integer M such that $(\forall m)(\forall n)(n > M \wedge m > M \Rightarrow |A_n - A_m| < \varepsilon)$; i.e., $\{A_n\}_{n=1}^{\infty} \subset \mathbb{C}$ is Cauchy. From the completeness of the complex numbers, if follows that $\{A_n\}_{n=1}^{\infty}$ is convergent to some complex number; let $\lim_{n \to \infty} A_n = A$.

We want to show that A is also equal to $\lim_{\substack{t \to w \\ t \in \Omega}} f(t)$. Again we suppose that $\varepsilon > 0$ is given. From $f_n \underset{\Omega}{\rightrightarrows} f$ there exists a positive integer M_1 such that $n > M_1$ implies that $\left|\underset{(6)}{\underline{\hspace{3cm}}}\right| < \frac{\varepsilon}{3}$ for all $t \in \Omega$, while the convergence of $\{A_n\}_{n=1}^{\infty}$ yields a positive integer M_2 such that $|A_n - A| < \frac{\varepsilon}{3}$ whenever $n > M_2$. Fix n such that $n > \max\{M_1, M_2\}$. Then, for all $t \in \Omega$,

$$|f(t) - f_n(t)| < \frac{\varepsilon}{3} \quad \text{and} \quad |A_n - A| < \frac{\varepsilon}{3}.$$

Since $\lim_{\substack{t \to w \\ t \in \Omega}} f_n(t) = A_n$, there exists a $\delta > 0$ such that

$$|f_n(t) - A_n| < \frac{\varepsilon}{3} \text{ for all } t \in (N_\delta(w) - \{w\}) \cap \Omega.$$

From the triangular inequality, for all $t \in \Omega$ *such that* $0 < d(t, w) < \delta$,

$$|f(t) - A| \leq \underset{(7)}{\underline{\hspace{8cm}}} < \varepsilon.$$

Therefore, $\underset{(8)}{\underline{\hspace{3cm}}}$. ■

Acceptable responses are: (1) uniformly Cauchy, (2) $n > M \wedge m > M$, (3) $t \in \Omega$, (4) $|f_n(t) - A_n| < \frac{\varepsilon}{3}$, (5) $f_n(t) - f_m(t)$, (6) $f(t) - f_n(t)$, (7) $|f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$, and (8) $\lim_{\substack{t \to w \\ t \in \Omega}} f(t) = A$.

Theorem 8.3.3 (The Uniform Limit of Continuous Functions) *Let* $\{f_n\}_{n=1}^{\infty}$ *denote a sequence of complex-valued functions that are continuous on a subset* Ω *of a metric space* (X, d). *If* $f_n \underset{\Omega}{\rightrightarrows} f$, *then* f *is continuous on* Ω.

Proof. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of complex-valued functions that are continuous on a subset Ω of a metric space (X, d). Then for each $\zeta \in \Omega$, $\lim_{t \to \zeta} f_n(t) = f_n(\zeta)$. Taking $A_n = f_n(\zeta)$ in Theorem 8.3.1 yields the claim. ■

Remark 8.3.4 *The contrapositive of Theorem 8.3.3 affords us a nice way of showing that we do not have uniform convergence of a given sequence of functions. Namely, if the limit of a sequence of complex-valued functions that are continuous on a subset* Ω *of a metric space is a function that is not continuous on* Ω, *we may immediately conclude that the convergence in not uniform. Be careful about the appropriate use of this: The limit function being continuous IS NOT ENOUGH to conclude that the convergence is uniform.*

The converse of Theorem 8.3.3 is false. For example, we know that $\left\{\frac{1}{nz}\right\}_{n=1}^{\infty}$ converges pointwise to the continuous function $f(z) = 0$ in $\mathbb{C} - \{0\}$ and the convergence is not uniform. The following result offers a list of criteria under which continuity of the limit of a sequence of real-valued continuous functions ensures that the convergence must be uniform.

Theorem 8.3.5 *Suppose that* Ω *is a compact subset of a metric space* (X, d) *and* $\{f_n\}_{n=1}^{\infty}$ *satisfies each of the following:*

(i) $\{f_n\}_{n=1}^{\infty}$ *is a sequence of real-valued functions that are continuous on* Ω*;*

(ii) $f_n \underset{\Omega}{\longrightarrow} f$ *and* f *is continuous on* Ω*; and*

(iii) $(\forall n)(\forall w)(n \in \mathbb{J} \wedge w \in \Omega \Rightarrow f_n(w) \geq f_{n+1}(w))$.

Then $f_n \underset{\Omega}{\rightrightarrows} f$.

Excursion 8.3.6 *Fill in what is missing in order to complete the following proof of Theorem 8.3.5.*

Proof. For $\{f_n\}_{n=1}^{\infty}$ satisfying the hypotheses, set $g_n = f_n - f$. Then, for each $n \in \mathbb{J}$, g_n is continuous on Ω and, for each $\zeta \in \Omega$, $\lim_{n\to\infty} g_n(\zeta) = \underset{(1)}{\underline{\qquad}}$. Since $f_n(w) \geq f_{n+1}(w)$ implies that $f_n(w) - f(w) \geq f_{n+1}(w) - f(w)$, we also have that $(\forall n)(\forall w)\left(n \in \mathbb{J} \wedge w \in \Omega \Rightarrow \underset{(2)}{\underline{\qquad\qquad\qquad}}\right)$.

To see that $g_n \underset{\Omega}{\rightrightarrows} 0$, suppose that $\varepsilon > 0$ is given. For each $n \in \mathbb{J}$, let

$$K_n = \{x \in \Omega : g_n(x) \geq \varepsilon\}.$$

Because Ω and $\mathbb{R}$ are metric spaces, g_n is continuous, and $\{w \in \mathbb{R} : w \geq \varepsilon\}$ is a closed subset of $\mathbb{R}$, by Corollary 5.2.16 to the Open Set Characterization of Continuous Functions, $\underset{(3)}{\underline{\qquad\qquad\qquad}}$. As a closed subset of a compact metric space, from Theorem 3.3.37, we conclude that K_n is $\underset{(4)}{\underline{\qquad\qquad\qquad}}$. If $x \in K_{n+1}$, then $g_{n+1}(x) \geq \varepsilon$ and $g_n(x) \geq g_{n+1}(x)$; it follows from the transitivity of $\geq$ that $\underset{(5)}{\underline{\qquad\qquad\qquad}}$. Hence, $x \in K_n$. Since x was arbitrary, $(\forall x)(x \in K_{n+1} \Rightarrow x \in K_n)$; i.e., $\underset{(6)}{\underline{\qquad\qquad\qquad}}$. Therefore, $\{K_n\}_{n=1}^{\infty}$ is a $\underset{(7)}{\underline{\qquad\qquad\qquad}}$ sequence of compact subsets of Ω. From Corollary 3.3.44 to Theorem 3.3.43, $((\forall n \in \mathbb{J})(K_n \neq \emptyset)) \Rightarrow \bigcap_{k\in\mathbb{J}} K_k \neq \emptyset$.

Suppose that $w \in \Omega$. Then $\lim_{n\to\infty} g_n(w) = 0$ and $\{g_n(x)\}$ decreasing yields the existence of a positive integer M such that $n > M$ implies that $0 \leq g_n(w) < \varepsilon$. In particular, $w \notin K_{M+1}$ from which it follows that $w \notin \bigcap_{n\in\mathbb{J}} K_n$. Because w was arbitrary, $(\forall w \in \Omega)\left(w \notin \bigcap_{n \in \mathbb{J}} K_n\right)$; i.e., $\bigcap_{n \in \mathbb{J}} K_n = \emptyset$. We conclude that there exists a positive integer P such that $K_P = \emptyset$. Hence, $K_n = \emptyset$ for all $\underset{(8)}{\underline{\hspace{2cm}}}$; that is, for all $n \geq P$, $\{x \in \Omega : g_n(x) \geq \varepsilon\} = \emptyset$. Therefore,

$$(\forall x)(\forall n)(x \in \Omega \wedge n > P \Rightarrow 0 \leq g_n(x) < \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we have that $g_n \underset{\Omega}{\rightrightarrows} 0$ which is equivalent to showing that $f_n \underset{\Omega}{\rightrightarrows} f$. ■

Acceptable responses are: (1) 0, (2) $g_n(w) \geq g_{n+1}(w)$, (3) K_n is closed, (4) compact, (5) $g_n(x) \geq \varepsilon$, (6) $K_{n+1} \subset K_n$, (7) nested, and (8) $n \geq P$.

Remark 8.3.7 *Since compactness was referred to several times in the proof of Theorem 8.3.5, it is natural to want to check that the compactness was really needed. The example offered by our author in order to illustrate the need is* $\left\{\frac{1}{1+nx}\right\}_{n=1}^{\infty}$ *in the segment* $(0, 1)$.

Our results concerning transmission of integrability and differentiability are for sequences of functions of real-valued functions on subsets of $\mathbb{R}$.

Theorem 8.3.8 (Integration of Uniformly Convergent Sequences) *Let α be a function that is (defined and) monotonically increasing on the interval $I = [a, b]$. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions such that*

$$(\forall n)(n \in \mathbb{J} \Rightarrow f_n \in \mathfrak{R}(\alpha) \text{ on } I)$$

and $f_n \underset{[a,b]}{\rightrightarrows} f$. Then $f \in \mathfrak{R}(\alpha)$ on I and

$$\int_a^b f(x)\, d\alpha(x) = \lim_{n \to \infty} \int_a^b f_n(x)\, d\alpha(x)$$

Excursion 8.3.9 *Fill in what is missing in order to complete the following proof of the Theorem.*

Proof. For each $n \in J$, let $\varepsilon_n = \sup\limits_{x \in I} |f_n(x) - f(x)|$. Then

$$f_n(x) - \varepsilon_n \leq f(x) \leq \underset{(1)}{\underline{\hspace{4cm}}} \text{ for } a \leq x \leq b$$

and if follows that

$$\int_a^b (f_n(x) - \varepsilon_n)\, d\alpha(x) \leq \underline{\int_a^b} f(x)\, d\alpha(x) \leq \overline{\int_a^b} f(x)\, d\alpha(x) \leq \int_a^b (f_n(x) + \varepsilon_n)\, d\alpha(x). \tag{8.3}$$

Properties of linear ordering yield that

$$0 \leq \overline{\int_a^b} f(x)\, d\alpha(x) - \underline{\int_a^b} f(x)\, d\alpha(x) \leq \int_a^b (f_n(x) + \varepsilon_n)\, d\alpha(x) - \underset{(2)}{\underline{\hspace{6cm}}}. \tag{8.4}$$

Because the upper bound in equation (8.4) is equivalent to $\underset{(3)}{\underline{\hspace{8cm}}}$, we conclude that $(\forall n \in \mathbb{J}) \left(0 \leq \overline{\int_a^b} f(x)\, d\alpha(x) - \underline{\int_a^b} f(x)\, d\alpha(x) \leq \underset{(4)}{\underline{\hspace{6cm}}} \right)$. By Theorem 8.2.7, $\varepsilon_n \to 0$ as $n \to \infty$. Since $\overline{\int_a^b} f(x)\, d\alpha(x) - \underline{\int_a^b} f(x)\, d\alpha(x)$ is constant, we conclude that $\underset{(5)}{\underline{\hspace{8cm}}}$. Hence $f \in \Re(\alpha)$.

Now, from equation 8.3, for each $n \in J$,

$$\int_a^b (f_n(x) - \varepsilon_n)\, d\alpha(x) \leq \int_a^b f(x)\, d\alpha(x) \leq \int_a^b (f_n(x) + \varepsilon_n)\, d\alpha(x).$$

(6) Finish the proof in the space provided.

■

Acceptable responses are:(1) $f_n(x)+\varepsilon_n$ (2) $\int_a^b (f_n(x)-\varepsilon_n)\,d\alpha(x)$, (3) $\int_a^b e\varepsilon_n d\alpha(x)$, (4) $2\varepsilon_n[\alpha(b)-\alpha(a)]$, (5) $\overline{\int_a^b} f(x)\,d\alpha(x)=\underline{\int_a^b} f(x)\,d\alpha(x)$, (6) Hopefully, you thought to repeat the process just illustrated. From the modified inequality it follows that $\left|\int_a^b f(x)\,d\alpha(x)-\int_a^b f_n(x)\,d\alpha(x)\right|\le\varepsilon_n[\alpha(b)-\alpha(a)]$; then because $\varepsilon_n\to 0$ as $n\to\infty$, given any $\varepsilon>0$ there exists a positive integer M such that $n>M$ implies that $\varepsilon_n[\alpha(b)-\alpha(a)]<\varepsilon$.

Corollary 8.3.10 *If $f_n\in\Re(\alpha)$ on $[a,b]$, for each $n\in\mathbb{J}$, and $\sum\limits_{k=1}^{\infty} f_k(x)$ converges uniformly on $[a,b]$ to a function f, then $f\in\Re(\alpha)$ on $[a,b]$ and*

$$\int_a^b f(x)\,d\alpha(x)=\sum_{k=1}^{\infty}\int_a^b f_k(x)\,d\alpha(x).$$

Having only uniform convergence of a sequence of functions is insufficient to make claims concerning the sequence of derivatives. There are various results that offer some additional conditions under which differentiation is transmitted. If we restrict ourselves to sequences of real-valued functions that are continuous on an interval $[a,b]$ and Riemann integration, then we can use the Fundamental Theorems of Calculus to draw analogous conclusions. Namely, we have the following two results.

Theorem 8.3.11 *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions that are continuous on the interval $[a,b]$ and $f_n \underset{[a,b]}{\rightrightarrows} f$. For $c\in[a,b]$ and each $n\in\mathbb{J}$, let*

$$F_n(x)\underset{def}{=}\int_c^x f_n(t)\,dt.$$

Then f is continuous on $[a,b]$ and $F_n \underset{[a,b]}{\rightrightarrows} F$ where

$$F(x)=\int_c^x f(t)\,dt.$$

The proof is left as an exercise.

Theorem 8.3.12 *Suppose that* $\{f_n\}_{n=1}^{\infty}$ *is such that* $f_n \underset{[a,b]}{\longrightarrow} f$ *and, for each* $n \in \mathbb{J}$, f_n' *is continuous on an interval* $[a, b]$. *If* $f_n' \underset{[a,b]}{\rightrightarrows} g$ *for some function* g *that is defined on* $[a, b]$, *then* g *is continuous on* $[a, b]$ *and* $f'(x) = g(x)$ *for all* $x \in [a, b]$.

Proof. Suppose that $\{f_n\}_{n=1}^{\infty}$ is such that $f_n \underset{[a,b]}{\longrightarrow} f$, f_n' is continuous on an interval $[a, b]$ for each $n \in \mathbb{J}$, and $f_n' \underset{[a,b]}{\rightrightarrows} g$ for some function g that is defined on $[a, b]$. From the Uniform Limit of Continuous Functions Theorem, g is continuous. Because each f_n' is continuous and $f_n' \underset{[a,b]}{\rightrightarrows} g$, by the second Fundamental Theorem of Calculus and Theorem 8.3.11, for $[c, x] \subset [a, b]$

$$\int_c^x g(t)\,dt = \lim_{n\to\infty} \int_c^x f_n'(t)\,dt = \lim_{n\to\infty} \left[f_n(x) - f_n(c)\right].$$

Now the pointwise convergence of $\{f_n\}$ yields that $\lim_{n\to\infty} \left[f_n(x) - f_n(c)\right] = f(x) - f(c)$. Hence, from the properties of derivative and the first Fundamental Theorem of Calculus, $g(x) = f'(x)$. ■

We close with the variation of 8.3.12 that is in our text; it is more general in that it does not require continuity of the derivatives and specifies convergence of the original sequence only at a point.

Theorem 8.3.13 *Suppose that* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of real-valued functions that are differentiable on an interval* $[a, b]$ *and that there exists a point* $x_0 \in [a, b]$ *such that* $\lim_{n\to\infty} f_n(x_0)$ *exists. If* $\left\{f_n'\right\}_{n=1}^{\infty}$ *converges uniformly on* $[a, b]$ *then* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly on* $[a, b]$ *to some function* f *and*

$$(\forall x)\left(x \in [a, b] \Rightarrow f'(x) = \lim_{n\to\infty} f_n'(x)\right).$$

Excursion 8.3.14 *Fill in what is missing in order to complete the following proof of Theorem 8.3.13.*

Proof. Suppose $\varepsilon > 0$ is given. Because $\{f_n(x_0)\}_{n=1}^{\infty}$ is convergent sequence of real numbers and $\mathbb{R}$ is complete, $\{f_n(x_0)\}_{n=1}^{\infty}$ is $\underset{(1)}{\underline{\hspace{3cm}}}$. Hence, there exists a positive integer M_1 such that $n > M_1$ and $m > M_1$ implies that

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}.$$

Because $\left\{ f_n' \right\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$, by Theorem $\underset{(2)}{\underline{\qquad\quad}}$, there exists a positive integer M_2 such that $n > M_2$ and $m > M_2$ implies that

$$\left| f_n'(\xi) - f_m'(\xi) \right| < \frac{\varepsilon}{2(b-a)} \text{ for } \underset{(3)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}}.$$

For fixed m and n, let $F = f_n - f_m$. Since each f_k is differentiable on $[a, b]$, F is differentiable on (a, b) and continuous on $[a, b]$. From the $\underset{(4)}{\underline{\qquad\qquad\qquad}}$ Theorem, for any $[x, t] \subset (a, b)$, there exists a $\xi \in (x, t)$ such that $F(x) - F(t) = F'(\xi)(x - t)$. Consequently, if $m > M_2$ and $n > M_2$, for any $[x, t] \subset (a, b)$, there exists a $\xi \in (x, t)$, it follows that

$$\begin{aligned} |(f_n(x) - f_m(x)) - (f_n(t) - f_m(t))| &= \left| f_n'(\xi) - f_m'(\xi) \right| |x - t| \\ &< \frac{\varepsilon}{2(b-a)} |x - t| \leq \underset{(5)}{\underline{\qquad\quad}}. \end{aligned} \tag{8.5}$$

Let $M = \max\{M_1, M_2\}$. Then $m > M$ and $n > M$ implies that

$$\begin{aligned} & |f_n(w) - f_m(w)| \\ \leq \quad & |(f_n(w) - f_m(w)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| < \varepsilon \end{aligned}$$

for any $w \in [a, b]$. Hence, $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$ to some function. Let f denote the limit function; i.e., $f(x) = \lim_{n \to \infty} f_n(x)$ for each $x \in [a, b]$ and $f_n \underset{[a,b]}{\rightrightarrows} f$.

Now we want to show that, for each $x \in [a, b]$, $f'(x) = \lim_{n \to \infty} f_n'(x)$; i.e., for fixed $x \in [a, b]$,

$$\lim_{n \to \infty} \lim_{t \to x} \frac{f_n(t) - f_n(x)}{t - x} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

where the appropriate one-sided limit is assumed when $x = a$ or $x = b$. To this end, for fixed $x \in [a, b]$, let

$$\phi_n(t) \underset{def}{=} \frac{f_n(t) - f_n(x)}{t - x} \quad and \quad \phi(t) \underset{def}{=} \frac{f(t) - f(x)}{t - x}$$

for $t \in [a, b] - \{x\}$ and $n \in \mathbb{J}$. Then, $x \in (a, b)$ implies that $\lim_{t \to x} \phi_n(t) = f_n'(x)$, while $x = a$ and $x = b$ yield that $\lim_{t \to a^+} \phi_n(t) = f_n'(a)$ and $\lim_{t \to b^-} \phi_n(t) = f_n'(b)$, respectively. Suppose $\varepsilon > 0$ if given. If $m > M_2$, $n > M_2$, and $t \in [a, b] - \{x\}$, then

$$|\phi_n(t) - \phi_m(t)| = \underset{(6)}{\underline{\hspace{6cm}}} < \frac{\varepsilon}{2(b-a)}$$

from equation (8.5). Thus, $\{\phi_n\}_{n=1}^{\infty}$ is uniformly Cauchy and, by Theorem 8.2.3, uniformly convergent on $t \in [a, b] - \{x\}$. Since $f(t) = \lim_{n\to\infty} f_n(t)$ for $t \in [a, b]$, we have that

$$\lim_{n\to\infty} \phi_n(t) = \phi(t).$$

Consequently, $\phi_n \rightrightarrows \phi$ on $[a, b] - \{x\}$. Finally, applying Theorem 8.3.1 to the sequence $\{\phi_n\}_{n=1}^{\infty}$, where $A_n = f_n'(x)$ yields that

$$f'(x) = \lim_{t\to x} \phi(t) = \underset{(7)}{\underline{\hspace{3cm}}}.$$

■

***Acceptable responses are: (1) Cauchy, (2) 8.2.3, (3) all $\xi \in [a, b]$, (4) Mean-Value, (5) $\frac{\varepsilon}{2}$,
(6) $\left|\frac{f_n(t) - f_n(x)}{t-x} - \frac{f_m(t) - f_m(x)}{t-x}\right| = \frac{|(f_n(t) - f_m(t)) - (f_n(x) - f_m(x))|}{|t-x|}$,
(7) $\lim_{n\to\infty} A_n = \lim_{n\to\infty} f_n'(x)$.***

Rudin ends the section of our text that corresponds with these notes by constructing an example of a real-valued continuous function that is nowhere differentiable.

Theorem 8.3.15 *There exists a real-valued function that is continuous on $\mathbb{R}$ and nowhere differentiable on $\mathbb{R}$.*

Proof. First we define a function ϕ that is continuous on $\mathbb{R}$, periodic with period 2, and not differentiable at each integer. To do this, we define the function in a interval that is "2 wide" and extend the definition by reference to the original part. For $x \in [-1, 1]$, suppose that $\phi(x) = |x|$ and, for all $x \in R$, let $\phi(x+2) = \phi(x)$.

In the space provided sketch a graph of ϕ.

The author shows that the function

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi\left(4^n x\right)$$

satisfies the needed conditions. Use the space provided to fill in highlights of the justification.

■

8.4 Families of Functions

Since any sequence of functions is also a set of functions, it is natural to ask questions about sets of functions that are related by some commonly shared nice behavior. The general idea is to seek additional properties that will shared by such sets of functions. For example, if $\mathcal{F}$ is the set of all real-valued functions from $[0, 1]$ into $[0, 1]$ that are continuous, we have seen that an additional shared property is that $(\forall f)(f \in \mathcal{F} \Rightarrow (\exists t)(t \in [0, 1] \wedge f(t) = t))$. In the last section, we considered sets of functions from a metric space into $\mathbb{C}$ or $\mathbb{R}$ and examined some of the consequences of uniform convergence of sequences.

Another view of sets of functions is considering the functions as points in a metric space. Let $\mathcal{C}([a,b])$ denote the family of real-valued functions that are continuous on the interval $I = [a,b]$. For f and g in $\mathcal{C}([a,b])$, we have seen that

$$\rho_\infty(f,g) = \max_{a \leq x \leq b} |f(x) - g(x)|$$

and

$$\rho(f,g) = \int_a^b |f(x) - g(x)|\, dx.$$

are metrics on $\mathcal{C}([a,b])$. As a homework problem (Problem Set H, #14), you will show that $(\mathcal{C}([a,b]), \rho)$ is not a complete metric space. On the other hand, $(\mathcal{C}([a,b]), \rho_\infty)$ is complete. In fact, the latter generalizes to the set of complex-valued functions that are continuous and bounded on the same domain.

Definition 8.4.1 *For a metric space (X,d), let $\mathcal{C}(X)$ denote the set of all complex-valued functions that are continuous and bounded on the domain X and, corresponding to each $f \in \mathcal{C}(X)$ the* ***supremum norm*** *or* ***sup norm*** *is given by*

$$\|f\| = \|f\|_X = \sup_{x \in X} |f(x)|.$$

It follows directly that $\|f\|_X = 0 \Leftrightarrow f(x) = 0$ for all $x \in X$ and

$$(\forall f)(\forall g)\left(f, g \in \mathcal{C}(X) \Rightarrow \|f+g\|_X \leq \|f\|_X + \|g\|_X\right).$$

The details of our proof for the corresponding set-up for $\mathcal{C}([a,b])$ allow us to claim that $\rho_\infty(f,g) = \|f-g\|_X$ is a metric for $\mathcal{C}(X)$.

Lemma 8.4.2 *The convergence of sequences in $\mathcal{C}(x)$ with respect to ρ_∞ is equivalent to uniform convergence of sequences of continuous functions in subsets of X.*

Use the space below to justify the claim made in the lemma.

Hopefully, you remembered that the metric replaces the occurrence of the absolute value (or modulus) is the statement of convergence. The immediate translation is that for every $\varepsilon > 0$, there exists a positive integer M such that $n > M$ implies that $\rho_\infty(f_n, f) < \varepsilon$. Of course, you don't want to stop there; the statement $\rho_\infty(f_n, f) < \varepsilon$ translates to $\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon$ which yields that $(\forall x)(x \in X \Rightarrow |f_n(x) - f(x)| < \varepsilon)$. This justifies that convergence of $\{f_n\}_{n=1}^\infty$ with respect to ρ_∞ implies that $\{f_n\}_{n=1}^\infty$ converges uniformly to f. Since the converse also follows immediately from the definitions, **we can conclude that convergence of sequences in $C(X)$ with respect to ρ_∞ is equivalent to uniform convergence**.

Theorem 8.4.3 *For a metric space X, $(\mathcal{C}(X), \rho_\infty)$ is a complete metric space.*

Excursion 8.4.4 *Fill in what is missing in order to complete the following proof of Theorem 8.4.3.*

Proof. Since $(\mathcal{C}(X), \rho_\infty)$ is a metric space, from Theorem 4.2.9, we know that any convergent sequence in $\mathcal{C}(X)$ is Cauchy.

Suppose that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $(\mathcal{C}(X), \rho_\infty)$ and that $\varepsilon > 0$ is given. Then there exists a positive integer M such that $n > M$ and $m > M$ implies that $\underset{(1)}{\underline{\hspace{6cm}}}$; i.e., for $n > M$ and $m > M$,

$$\sup_{\xi \in X} |f_n(\xi) - f_m(\xi)| < \varepsilon.$$

Hence, $(\forall x)\left(x \in X \Rightarrow \underset{(2)}{\underline{\hspace{3cm}}}\right)$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\{f_n\}_{n=1}^\infty$ is $\underset{(3)}{\underline{\hspace{6cm}}}$. As a $\underset{(3)}{\underline{\hspace{6cm}}}$

sequence of complex-valued functions on a metric space X, by Theorem $\underset{(4)}{\underline{\hspace{1.5cm}}}$, $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent. Let $f : X \to \mathbb{C}$ denote the uniform limit. Because $f_n \underset{X}{\rightrightarrows} f$, for any $\varepsilon > 0$ there exists a positive integer M such that $n > M$ implies that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \text{ for all } x \in X.$$

In particular, $\underset{(5)}{\underline{\hspace{3cm}}} = \sup_{\xi \in X} |f_n(\xi) - f(\xi)| \leq \frac{\varepsilon}{2} < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\rho_\infty(f_n, f) \to 0$ as $n \to \infty$. Hence, $\{f_n\}_{n=1}^{\infty}$ is convergent to f in $(\mathcal{C}(X), \rho_\infty)$.

Now we want to show that $f \in \mathcal{C}(X)$. As the uniform limit of continuous functions from a metric space X in $\mathbb{C}$, we know that f is $\underset{(6)}{\underline{\hspace{3cm}}}$. Because $f_n \underset{X}{\rightrightarrows} f$, corresponding to $\varepsilon = 1$ there exists a positive integer M such that $n > M$ implies that $|f_n(x) - f(x)| < 1$ for all $x \in X$. In particular, from the (other) triangular inequality, we have that

$$(\forall x)(x \in X \Rightarrow |f(x)| < |f_{M+1}(x)| + 1). \tag{8.6}$$

Since $f_{M+1} \in \mathcal{C}(X)$, f_{M+1} is continuous and $\underset{(7)}{\underline{\hspace{3cm}}}$ on X. From equation (8.6), it follows that f is $\underset{(7)}{\underline{\hspace{3cm}}}$ on X. Because $f : X \to \mathbb{C}$ is continuous and bounded on X, $\underset{(8)}{\underline{\hspace{3cm}}}$. The sequence $\{f_n\}_{n=1}^{\infty}$ was an arbitrary Cauchy sequence in $(\mathcal{C}(X), \rho_\infty)$. Consequently, we conclude that every Cauchy sequence in $(\mathcal{C}(X), \rho_\infty)$ is convergent in $(\mathcal{C}(X), \rho_\infty)$. This concludes that proof that convergence in $(\mathcal{C}(X), \rho_\infty)$ is equivalent to being Cauchy in $(\mathcal{C}(X), \rho_\infty)$. ■

Acceptable responses are: (1) $\rho_\infty(f_n, f_m) < \varepsilon$, (2) $|f_n(x) - f_m(x)| < \varepsilon$, (3) uniformly Cauchy, (4) 8.2.3, (5) $\rho_\infty(f_n, f)$, (6) continuous, (7) bounded, and (8) $f \in \mathcal{C}(X)$.

Remark 8.4.5 *At first, one might suspect that completeness is an intrinsic property of a set. However, combining our prior discussion of the metric spaces* $(\mathbb{R}, d)$ *and* $(\mathbb{Q}, d)$ *where d denotes the Euclidean metric with our discussion of the two metrics* *on $\mathcal{C}([a,b])$ leads us to the conclusion that completeness depends on two things: the nature of the underlying set and the way in which distance is measured on the set.*

We have made a significant transition from concentration on sets whose elements are points on a plane or number line (or Euclidean n-space) to sets where the points are functions. Now that we have seen a setting that gives us the notion of completeness in this new setting, it is natural to ask about generalization or transfer of other general properties. What might characterizations of compactness look like? Do we have an analog for the Bolzano-Weierstrass Theorem? In this discussion, we will concentrate on conditions that allow us to draw conclusions concerning sequences of bounded functions and subsequences of convergent sequences. We will note right away that care must be taken.

Definition 8.4.6 *Let $\mathcal{F}$ denote a family of complex-valued functions defined on a metric space (Ω, d). Then*

(a) *$\mathcal{F}$ is said to be* ***uniformly bounded*** *on Ω if and only if*

$$(\exists M \in \mathbb{R})\,(\forall f)\,(\forall w)\,(f \in \mathcal{F} \wedge w \in \Omega \Rightarrow |f(w)| \leq M).$$

(b) *$\mathcal{F}$ is said to be* ***locally uniformly bounded*** *on Ω if and only if*

$$(\forall w)\,(w \in \Omega \Rightarrow (\exists N_w)\,(N_w \subset \Omega \wedge \mathcal{F} \text{ is uniformly bounded on } N_w)).$$

(c) *any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ is said to be* ***pointwise bounded*** *on Ω if and only if*

$(\forall w)\left(w \in \Omega \Rightarrow \{f_n(w)\}_{n=1}^{\infty} \text{ is bounded}\right)$; i.e., corresponding to each $w \in \Omega$, there exists a positive real number $M_w \underset{def}{=} \phi(w)$ such that

$$|f_n(w)| < M_w \text{ for all } n \in \mathbb{J}.$$

Example 8.4.7 *For $x \in \Omega = \mathbb{R} - \{0\}$, let $\mathcal{F} = \left\{ f_n(x) = \dfrac{x}{n^2 + x^2} : n \in \mathbb{J} \right\}$. Then, for $w \in \Omega$, taking $\phi(w) = \dfrac{2|w|}{1 + w^2}$ implies that $|f_n(w)| < \phi(w)$ for all $n \in \mathbb{J}$. Thus, $\mathcal{F}$ is pointwise bounded on Ω.*

Remark 8.4.8 *Uniform boundedness of a family implies that each member of the family is bounded but not conversely.*

Excursion 8.4.9 *Justify this point with a discussion of* $\mathcal{F} = \{f_n(z) = nz : n \in \mathbb{J}\}$ *on* $U_r = \{z \in \mathbb{C} : |z| < r\}$.

Hopefully, you observed that each member of $\mathcal{F}$ is bounded in U_r but no single bound works for all of the elements in $\mathcal{F}$.

Remark 8.4.10 *Uniform boundedness of a family implies local uniform boundedness but not conversely.*

Excursion 8.4.11 *To see this, show that* $\left\{\dfrac{1}{1-z^n} : n \in \mathbb{J}\right\}$ *is locally uniformly bounded in* $U = \{z : |z| < 1\}$ *but not uniformly bounded there.*

Neighborhoods that can justify local uniform boundedness vary; the key is to capitalize on the fact that you can start with an arbitrary fixed $z \in U$ and make use of its distance from the origin to define a neighborhood. For example, given $z_0 \in U$ with $|z_0| = r < 1$, let $N_{z_0} = N\left(z_0, \dfrac{1-r}{4}\right)$; now, $N_{z_0} \subset U$ and $\left|(1-z^n)^{-1}\right| \leq (1-|z|)^{-1}$ can be used to justify that, for each $n \in \mathbb{J}$, $\left|(1-z^n)^{-1}\right| < 4(1-r)/3$. The latter allows us to conclude that the given family is uniformly bounded on N_{z_0}. Since z_0 was arbitrary, we can claim local uniform boundedness in U. One way to justify the lack of uniform boundedness is to investigate the behavior of the functions in the family at the points $\sqrt[n]{1-n^{-1}}$.

The following theorem gives us a characterization for local uniform boundedness when the metric space is a subset of $\mathbb{R}$ or $\mathbb{C}$.

Theorem 8.4.12 *A family of complex valued functions $\mathcal{F}$ on a subset Ω of $\mathbb{C}$ is locally uniformly bounded in Ω if and only if $\mathcal{F}$ is uniformly bounded on every compact subset of Ω.*

Proof. ($\Leftarrow$) This is an immediate consequence of the observation that the closure of a neighborhood in $\mathbb{C}$ or $\mathbb{R}$ is compact.

($\Rightarrow$) Suppose $\mathcal{F}$ is locally uniformly bounded on a domain Ω and K is a compact subset of Ω. Then, for each $z \in K$ there exists a neighborhood of z, $N(z;\epsilon_z)$ and a positive real number, M_z, such that

$$|f(\zeta)| \leq M_z, \text{ for all } \zeta \in N(z;\epsilon_z).$$

Since $\{N(z;\epsilon_z) : z \in K\}$ covers K, we know that there exists a finite subcover, say $\left\{N\left(z_j;\epsilon_{z_j}\right) : j = 1, 2, \cdots, n\right\}$. Then, for $M = \max\left\{M_{z_j} : 1 \leq j \leq n\right\}$, $|f(z)| \leq M$, for all $z \in K$, and we conclude that $\mathcal{F}$ is uniformly bounded on K. ■

Remark 8.4.13 *Note that Theorem 8.4.12 made specific use of the Heine-Borel Theorem; i.e., the fact that we were in a space where compactness is equivalent to being closed and bounded.*

Remark 8.4.14 *In our text, an example is given to illustrate that a uniformly bounded sequence of real-valued continuous functions on a compact metric space need not yield a subsequence that converges (even) pointwise on the metric space. Because the verification of the claim appeals to a theorem given in Chapter 11 of the text, at this point we accept the example as a reminder to be cautious.*

Remark 8.4.15 *Again by way of example, the author of our text illustrates that it is not the case that every convergent sequence of functions contains a uniformly convergent subsequence. We offer it as our next excursion, providing space for you to justify the claims.*

Excursion 8.4.16 *Let* $\Omega = \{x \in \mathbb{R} : 0 \leq x \leq 1\} = [0,1]$ *and*

$$\mathcal{F} = \left\{ f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2} : n \in \mathbb{J} \right\}.$$

(a) *Show that* $\mathcal{F}$ *is uniformly bounded in* $\mathbf{\Omega}$.

(b) *Find the pointwise limit of* $\{f_n\}_{n=1}^{\infty}$ *for* $x \in \mathbf{\Omega}$.

(c) *Justify that no subsequence of* $\{f_n\}_{n=1}^{\infty}$ *can converge uniformly on* $\mathbf{\Omega}$.

For (a), observing that $x^2 + (1 - nx)^2 \geq x^2 > 0$ for $x \in (0, 1]$ and $f_n(0) = 0$ for each $n \in \mathbb{J}$ yields that $|f_n(x)| \leq 1$ for $x \in \mathbf{\Omega}$. In (b), since the only occurrence of n is in the denominator of each f_n, for each fixed $x \in \mathbf{\Omega}$, the corresponding sequence of real goes to 0 as $n \to \infty$. For (c), in view of the negation of the definition of uniform convergence of a sequence, the behavior of the sequence $\{f_n\}_{n=1}^{\infty}$ at the points $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ allows us to conclude that no subsequences of $\{f_n\}_{n=1}^{\infty}$ will converge uniformly on $\mathbf{\Omega}$.

Now we know that we don't have a "straight" analog for the Bolzano-Weierstrass Theorem when we are in the realm of families of functions in $\mathcal{C}(X)$. This poses the challenge of finding an additional property (or set of properties) that will yield such an analog. Towards that end, we introduce define a property that requires "local and global" uniform behavior over a family.

Definition 8.4.17 *A family $\mathcal{F}$ of complex-valued functions defined on a metric space (Ω, d) is **equicontinuous** on Ω if and only if*

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall f)\,(\forall u)\,(\forall v)\,(f \in \mathcal{F} \wedge u \in \Omega \wedge v \in \Omega \wedge d(u, v) < \delta \\ \Rightarrow |f(u) - f(v)| < \varepsilon).$$

Remark 8.4.18 *If $\mathcal{F}$ is equicontinuous on Ω, then each $f \in \mathcal{F}$ is clearly uniformly continuous in Ω.*

Excursion 8.4.19 *On the other hand, for $U_R = \{z : |z| \leq R\}$, show that each function in $\mathcal{F} = \{nz : n \in \mathbb{J}\}$ is uniformly continuous on U_R though $\mathcal{F}$ is not equicontinuous on U_R.*

Excursion 8.4.20 *Use the Mean-Value Theorem to justify that*

$$\left\{ f_n(x) = n \sin \frac{x}{n} : n \in \mathbb{J} \right\}$$

is equicontinuous in $\Omega = [0, \infty)$

The next result is particularly useful when we can designate a denumerable subset of the domains on which our functions are defined. When the domain is an open connected subset of $\mathbb{R}$ or $\mathbb{C}$, then the rationals or points with the real and imaginary parts as rational work very nicely. In each case the denumerable subset is dense in the set under consideration.

Lemma 8.4.21 *If $\{f_n\}_{n=1}^{\infty}$ is a pointwise bounded sequence of complex-valued functions on a denumerable set E, then $\{f_n\}_{n=1}^{\infty}$ has a subsequence $\left\{f_{n_k}\right\}_{k=1}^{\infty}$ that converges pointwise on E.*

Excursion 8.4.22 *Finish the following proof.*

Proof. Let $\{f_n\}$ be sequence of complex-valued functions that is pointwise bounded on a denumerable set E. Then the set E can be realized as a sequence$\{w_k\}$ of distinct points. This is a natural setting for application of the Cantor diagonalization process that we saw earlier in the proof of the denumerability of the rationals. From the Bolzano–Weierstrass Theorem, $\{f_n(w_1)\}$ bounded implies that there exists a convergent subsequence $\left\{f_{n,1}(w_1)\right\}$. The process can be applied to $\left\{f_{n,1}(w_2)\right\}$ to obtain a subsequence $\left\{f_{n,2}(w_2)\right\}$ that is convergent.

$$\begin{array}{cccc} f_{1,1} & f_{2,1} & f_{3,1} & \cdots \\ f_{1,2} & f_{2,2} & f_{3,2} & \cdots \\ & \cdot\ \cdot\ \cdot & & \\ & \cdot\ \cdot\ \cdot & & \end{array}$$

In general, $\left\{f_{n,j}\right\}_{n=1}^{\infty}$ is such that $\left\{f_{n,j}\left(w_j\right)\right\}_{n=1}^{\infty}$ is convergent and $\left\{f_{n,j}\right\}_{n=1}^{\infty}$ is a subsequence of each of $\left\{f_{n,k}\right\}_{n=1}^{\infty}$ for $k = 1, 2, ..., j-1$. Now consider $\left\{f_{n,n}\right\}_{n=1}^{\infty}$

■

*** For $x \in E$, there exists an $M \in \mathbb{J}$ such that $x = w_M$. Then $\left\{f_{n,n}\right\}_{n=M+1}^{\infty}$ is a subsequence of $\left\{f_{n,M}\right\}_{n=M+1}^{\infty}$ from which it follows that $\left\{f_{n,n}(x)\right\}$ is convergent at x. ***

The next result tells us that if we restrict ourselves to domains K that are compact metric spaces that any uniformly convergent sequence in $\mathcal{C}(K)$ is also an equicontinuous family.

Theorem 8.4.23 *Suppose that* (K, d) *is a compact metric space and the sequence of functions* $\{f_n\}_{n=1}^{\infty}$ *is such that* $(\forall n)\,(n \in \mathbb{J} \Rightarrow f_n \in \mathcal{C}(K))$. *If* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly on* K, *then* $\mathcal{F} = \{f_n : n \in \mathbb{J}\}$ *is equicontinuous on* K.

Proof. Suppose that (K, d) is a compact metric space, the sequence of functions $\{f_n\}_{n=1}^{\infty} \subset \mathcal{C}(K)$ converges uniformly on K and $\varepsilon > 0$ is given. By Theorem 8.2.3, $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy on K. Thus, there exists a positive integer M such that $n \geq M$ implies that $\|f_n - f_m\|_K < \dfrac{\varepsilon}{3}$. In particular,

$$\|f_n - f_M\|_K < \frac{\varepsilon}{3} \text{ for all } n > M.$$

Because each f_n is continuous on a compact set, from the Uniform Continuity Theorem, for each $n \in \mathbb{J}$, f_n is uniformly continuous on K. Hence, for each $j \in \{1, 2, ..., M\}$, there exists a $\delta_j > 0$ such that $x, y \in K$ and $d(x, y) < \delta_j$ implies that $\left|f_j(x) - f_j(y)\right| < \dfrac{\varepsilon}{3}$. Let $\delta = \min\limits_{1 \leq j \leq M} \delta_j$. Then

$$(\forall j)(\forall x)(\forall y)\left[\begin{array}{c}(j \in \{1, 2, ..., M\} \wedge x, y \in K \wedge d(x, y) < \delta) \Rightarrow \\ \left|f_j(x) - f_j(y)\right| < \dfrac{\varepsilon}{3}\end{array}\right]. \tag{8.7}$$

For $n > M$ and $x, y \in K$ such that $d(x, y) < \delta$, we also have that

$$\begin{aligned}|f_n(x) - f_n(y)| \leq |f_n(x) - f_M(x)| &+ \\ |f_M(x) - f_M(y)| + |f_M(y) - f_n(y)| &< \varepsilon.\end{aligned} \tag{8.8}$$

From (8.7) and (8.8) and the fact that $\varepsilon > 0$ was arbitrary, we conclude that

$$\begin{aligned}(\forall \varepsilon > 0)(\exists \delta > 0)(\forall f_n)(\forall u)(\forall v)\,(f_n \in \mathcal{F} \wedge u, v \in K \wedge d(u, v) < \delta \\ \Rightarrow |f(u) - f(v)| < \varepsilon); \text{ i.e.,}\end{aligned}$$

$\mathcal{F}$ is equicontinuous on K. ■

We are now ready to offer conditions on a subfamily of $\mathcal{C}(K)$ that will give us an analog to the Bolzano-Weierstrass Theorem.

Theorem 8.4.24 *Suppose that* (K, d) *is a compact metric space and the sequence of functions* $\{f_n\}_{n=1}^{\infty}$ *is such that* $(\forall n)\,(n \in \mathbb{J} \Rightarrow f_n \in \mathcal{C}(K))$. *If* $\{f_n : n \in \mathbb{J}\}$ *is pointwise bounded and equicontinuous on* K, *then*

(a) $\{f_n : n \in \mathbb{J}\}$ *is uniformly bounded on* K *and*

(b) $\{f_n\}_{n=1}^{\infty}$ *contains a subsequence that is uniformly convergent on* K.

Excursion 8.4.25 *Fill in what is missing in order to complete the following proof of Theorem 8.4.24.*

Proof. Suppose that (K, d) is a compact metric space, the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is such that $(\forall n)\,(n \in \mathbb{J} \Rightarrow f_n \in \mathcal{C}(K))$, the family $\{f_n : n \in \mathbb{J}\}$ is pointwise bounded and equicontinuous on K.

Proof of part (a):

Let $\varepsilon > 0$ be given. Since $\{f_n : n \in \mathbb{J}\}$ is equicontinuous on K, there exists a $\delta > 0$ such that

$$(\forall n)\,(\forall x)\,(\forall y)\left[(n \in \mathbb{J} \wedge x, y \in K \wedge d(x, y) < \delta) \Rightarrow |f_n(x) - f_n(y)| < \varepsilon\right]. \tag{8.9}$$

Because $\{N_\delta(u) : u \in K\}$ forms an $\underset{(1)}{\underline{\hspace{3cm}}}$ for K and K is compact, there exists a finite number of points, say $p_1, p_2, \ldots, p_k$, such that $K \subset \underset{(2)}{\underline{\hspace{3cm}}}$. On the other hand, $\{f_n : n \in \mathbb{J}\}$ is pointwise bounded; consequently, for each p_j, $j \in \{1, 2, \ldots, k\}$, there exists a positive real number M_j such that

$$(\forall n)\left(n \in \mathbb{J} \Rightarrow \left|f_n\left(p_j\right)\right| < M_j\right).$$

For $M = \underset{(3)}{\underline{\hspace{3cm}}}$, it follows that

$$(\forall n)\,(\forall j)\left((n \in \mathbb{J} \wedge j \in \{1, 2, \ldots, k\}) \Rightarrow \left|f_n\left(p_j\right)\right| < M\right). \tag{8.10}$$

Suppose that $x \in K$. Since $K \subset \bigcup_{j=1}^{k} N_\delta\left(p_j\right)$ there exists an $m \in \{1, 2, \ldots, k\}$ such that $\underset{(4)}{\underline{\hspace{3cm}}}$. Hence, $d(x, p_m) < \delta$ and, from (8.9), we conclude that $\underset{(5)}{\underline{\hspace{3cm}}}$ for all $n \in \mathbb{J}$. But then $|f_n(x)| - |f_n(p_m)| \leq |f_n(x) - f_n(p_m)|$ yields that $|f_n(x)| < |f_n(p_m)| + \varepsilon$ for $\underset{(6)}{\underline{\hspace{3cm}}}$. From (8.10), we conclude that $|f_n(x)| < M + \varepsilon$ for all $n \in \mathbb{J}$. Since x was arbitrary, it follows that

$$(\forall n)\,(\forall x)\left[(n \in \mathbb{J} \wedge x \in K) \Rightarrow |f_n(x)| < M + \varepsilon\right]; \text{ i.e.,}$$

$\{f_n : n \in \mathbb{J}\}$ is $\underset{(7)}{\underline{\hspace{3cm}}}$.

Almost a proof of part (b):

If K were finite, we would be done. For K infinite, let E be a denumerable subset of K that is dense in K. (The reason for the "Almost" in the title of this part of the proof is that we did not do the Exercise #25 on page 45 for homework. If $K \subset \mathbb{R}$ or $K \subset \mathbb{C}$, then the density of the rationals leads immediately to a set E that satisfies the desired property; in the general case of an arbitrary metric space, Exercise #25 on page 45 indicates how we can use open coverings with rational radii to obtain such a set.) Because $\{f_n : n \in \mathbb{J}\}$ is $\underset{(8)}{\underline{\hspace{3cm}}}$ on E, by Lemma 8.4.21, there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$, say $\left\{g_j\right\}_{j=1}^{\infty}$, that is convergent for each $x \in E$.

Suppose that $\varepsilon > 0$ is given. Since $\{f_n : n \in \mathbb{J}\}$ is equicontinuous on K, there exists a $\delta > 0$ such that

$$(\forall n)\,(\forall x)\,(\forall y)\left[(n \in \mathbb{J} \wedge x, y \in K \wedge d\,(x, y) < \delta) \Rightarrow |f_n\,(x) - f_n\,(y)| < \frac{\varepsilon}{3}\right].$$

Because E is dense in K, $\{N_\delta\,(u) : u \in E\}$ forms an open cover for K. Because K is compact, we conclude that there exists a finite number of elements of E, say $w_1, w_2, ..., w_q$, such that

$$K \subset \bigcup_{j=1}^{q} N_\delta\left(w_j\right). \tag{8.11}$$

Since $\left\{w_1, w_2, ..., w_q\right\} \subset E$ and $\left\{g_j\,(x)\right\}_{j=1}^{\infty}$ is a convergent sequence of complex numbers for each $x \in E$, the completeness of $\mathbb{C}$, yields Cauchy convergence of $\left\{g_j\,(w_s)\right\}_{j=1}^{\infty}$for each w_s, $s \in \{1, 2, ..., q\}$. Hence, for each $s \in \{1, 2, ..., q\}$, there exists a positive integer M_s such that $n > M_s$ and $m > M_s$ implies that

$$|g_n\,(w_s) - g_m\,(w_s)| < \frac{\varepsilon}{3}.$$

Suppose that $x \in K$. From (8.11), there exists an $s \in \{1, 2, ..., q\}$ such that $\underset{(9)}{\underline{\hspace{3cm}}}$. Then $d\,(x, w_s) < \delta$ implies that

$$|f_n\,(x) - f_n\,(w_s)| < \frac{\varepsilon}{3}$$

for all $n \in \mathbb{J}$. Let $M = \max\{M_s : s \in \{1, 2, ..., q\}\}$. It follows that, for $n > M$ and $m > M$,

$$\begin{aligned}|g_n(x) - g_m(x)| &\leq |g_n(x) - g_n(w_s)| + \\ &\quad \left|\underset{(10)}{\underline{\qquad\qquad\qquad}}\right| + |g_m(w_s) - g_m(x)| < \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ and $x \in K$ were arbitrary, we conclude that

$$(\forall \varepsilon > 0)\,(\exists M \in \mathbb{J})\left[n > M \wedge m > M \Rightarrow (\forall x)\,(x \in K \Rightarrow |g_n(x) - g_m(x)| < \varepsilon)\right];$$

i.e., $\{g_j\}_{j=1}^{\infty}$ is $\underset{(11)}{\underline{\qquad\qquad\qquad}}$. By Theorem 8.4.23, $\{g_j\}_{j=1}^{\infty}$ is uniformly convergent on K as needed. ■

Acceptable responses are: (1) open cover, (2) $\bigcup_{j=1}^{k} N_\delta(p_j)$, (3) $\max\{M_j : j = 1, 2, ..., k\}$, (4) $N_\delta(p_m)$, (5) $|f_n(x) - f_n(p_m)| < \varepsilon$, (6) all $n \in \mathbb{J}$, (7) uniformly bounded on K, (8) pointwise bounded on K, (9) $x \in N_\delta(w_s)$, (10) $|g_n(w_s) - g_m(w_s)|$, (11) uniformly Cauchy on K.

Since we now know that for families of functions it is not the case that every convergent sequence of functions contains a uniformly convergent subsequence, families that do have that property warrant a special label.

Definition 8.4.26 *A family $\mathcal{F}$ of complex-valued functions defined on a metric space Ω is said to be* ***normal*** *in Ω if and only if every sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence $\{f_{n_k}\}$ that converges uniformly on compact subsets of Ω.*

Remark 8.4.27 *In view of Theorem 8.4.24, any family that is pointwise bounded and equicontinuous on a compact metric space K is normal in K.*

Our last definition takes care of the situation when the limits of the sequences from a family are in the family.

Definition 8.4.28 *A normal family of complex-valued functions $\mathcal{F}$ is said to be* ***compact*** *if and only if the uniform limits of all sequences converging in $\mathcal{F}$ are also members of $\mathcal{F}$.*

8.5 The Stone-Weierstrass Theorem

In view of our information concerning the transmission of nice properties of functions in sequences (and series), we would like to have results that enable us to realize a given function as the uniform limit of a sequence of nice functions. The last result that we will state in this chapter relates a given function to a sequence of polynomials. Since polynomials are continuously differentiable functions the theorem is particularly good news. We are offering the statement of the theorem without discussing the proof. Space is provided for you to insert a synopsis or comments concerning the proof that is offered by the author of our text on pages 159-160.

Theorem 8.5.1 *If* $f \in \mathcal{C}([a,b])$ *for* $a < b$, *then there exists a sequence of polynomials* $\{P_n\}_{n=1}^{\infty} \subset \mathcal{C}([a,b])$ *such that* $\lim_{n\to\infty} P_n(x) = f(x)$ *where the convergence is uniform of* $[a,b]$. *If* f *is a real-valued function then the polynomials can be taken as real.*

Space for Comments.

8.6 Problem Set H

1. Use properties of limits to find the pointwise limits for the following sequences of complex-valued functions on $\mathbb{C}$.

   (a) $\left\{\dfrac{nz}{1+nz^2}\right\}_{n=1}^{\infty}$

(b) $\left\{\dfrac{nz^2}{z+3n}\right\}_{n=1}^{\infty}$

(c) $\left\{\dfrac{z^n}{1+z^n}\right\}_{n=1}^{\infty}$

(d) $\left\{\dfrac{n^2z}{1+n^3z^2}\right\}_{n=1}^{\infty}$

(e) $\left\{\dfrac{1+n^2z}{1-n^2z}+\dfrac{n}{1+2n}\right\}_{n=1}^{\infty}$

(f) $\left\{ze^{-n|z|}\right\}_{n=1}^{\infty}$

2. For each $n \in \mathbb{J}$, let $f_n(x) = \dfrac{nx}{e^{nx}}$. Use the definition to prove that $\{f_n\}_{n=1}^{\infty}$ is pointwise convergent on $[0,\infty)$, uniformly convergent on $[\alpha,\infty)$ for any fixed positive real number α, and not uniformly convergent on $(0,\infty)$.

3. For each of the following sequences of real-valued functions on $\mathbb{R}$, use the definition to show that $\{f_n(x)\}_{n=1}^{\infty}$ converges pointwise to the specified $f(x)$ on the given set I; then determine whether or not the convergence is uniform. Use the definition or its negation to justify your conclusions concerning uniform convergence.

(a) $\{f_n(x)\}_{n=1}^{\infty} = \left\{\dfrac{2x}{1+nx}\right\}$; $f(x)=0$; $I=[0,1]$

(b) $\{f_n(x)\}_{n=1}^{\infty} = \left\{\dfrac{\cos nx}{\sqrt{n}}\right\}$; $f(x)=0$; $I=[0,1]$

(c) $\{f_n(x)\}_{n=1}^{\infty} = \left\{\dfrac{n^3x}{1+n^4x}\right\}$; $f(x)=0$; $I=[0,1]$

(d) $\{f_n(x)\}_{n=1}^{\infty} = \left\{\dfrac{n^3x}{1+n^4x^2}\right\}$; $f(x)=0$; $I=[a,\infty)$ where a is a positive fixed real number

(e) $\{f_n(x)\}_{n=1}^{\infty} = \left\{\dfrac{1-x^n}{1-x}\right\}$; $f(x)=\dfrac{1}{1-x}$; $I=\left[-\dfrac{1}{2},\dfrac{1}{2}\right]$

(f) $\{f_n(x)\}_{n=1}^{\infty} = \left\{nxe^{-nx^2}\right\}$; $f(x)=0$; $I=[0,1]$

4. For the sequence $\{f_n\}_{n=1}^{\infty}$ of real-valued functions on $\mathbb{R}$ given by $f_n(x) = \dfrac{(n+1)(n+2)x^n}{1-x}$ for $n \in \mathbb{J}$ and $f(x) = 0$ for $x \in I = [0,1]$, show that $f_n(x) \longrightarrow f(x)$ as $n \to \infty$ for each $x \in I$. Is is true that
$$\int_0^1 f_n(x)\,dx \longrightarrow \int_0^1 f(x)\,dx \text{ as } n \to \infty?$$

5. Suppose that the sequences of functions $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ converge uniformly to f and g, respectively, on a set A in a metric space (S,d). Prove that the sequence $\{f_n + g_n\}_{n=1}^{\infty}$ converges uniformly to $f+g$.

6. Determine all the values of h such that $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{\left(1+nx^2\right)\sqrt{n}}$ is uniformly convergent in $I = \{x \in \mathbb{R} : |x| < h\}$. (Hint: Justify that each $f_n(x) = \dfrac{x^2}{\left(1+nx^2\right)}$ is increasing as a function x and make use that the obtain an upper bound on the summand.)

7. Prove that, if $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n \cos nx$ converges uniformly for all $x \in \mathbb{R}$.

8. Suppose that $\displaystyle\sum_{n=1}^{\infty} n\,|b_n|$ is convergent and let $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin nx$ for $x \in \mathbb{R}$. Show that
$$f'(x) = \sum_{n=1}^{\infty} nb_n \cos nx$$
and that both $\displaystyle\sum_{n=1}^{\infty} b_n \sin nx$ and $\displaystyle\sum_{n=1}^{\infty} nb_n \cos nx$ converge uniformly for all $x \in \mathbb{R}$.

9. Prove that if a sequence of complex-valued functions on $\mathbb{C}$ converges uniformly on a set A and on a set B, then it converges uniformly on $A \cup B$.

10. Prove that if the sequence $\{f_n\}_{n=1}^{\infty}$ of complex-valued functions on $\mathbb{C}$ is uniformly convergent on a set Ω to a function f that is bounded on Ω, then

there exists a positive real number K and a positive integer M such that $(\forall n)(\forall x)(n > M \wedge x \in \Omega \Rightarrow |f_n(x)| < K)$.

11. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions each of which is continuous on an interval $I = [a,b]$. If $\{f_n\}_{n=1}^{\infty}$ is uniformly continuous on I, prove that there exists a positive real number K such that

    $(\forall n)(\forall x)(n \in J \wedge x \in I \Rightarrow |f_n(x)| < K)$.

12. Without appeal to Theorem 8.3.8; i.e., using basic properties of integrals, prove Theorem 8.3.11: Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions that are continuous on the interval $[a,b]$ and $f_n \underset{[a,b]}{\rightrightarrows} f$. For $c \in [a,b]$ and each $n \in \mathbb{J}$, let $F_n(x) \underset{def}{=} \int_c^x f_n(t)\,dt$.Then f is continuous on $[a,b]$ and $F_n \underset{[a,b]}{\rightrightarrows} F$ where $F(x) = \int_c^x f(t)\,dt$.

13. Compare the values of the integrals of the nth partials sums over the interval $[0,1]$ with the integral of their their limit in the case where $\sum_{k=1}^{\infty} f_k(x)$ is such that

$$f_1(x) = \begin{cases} x+1 & , \quad -1 \le x \le 0 \\ -x+1 & , \quad 0 < x \le 1 \end{cases},$$

and, for each $n = 2, 3, 4, ...$,

$$S_n(x) = \begin{cases} 0 & , \quad -1 \le x < \dfrac{-1}{n} \\ n^2x + n & , \quad \dfrac{-1}{n} \le x \le 0 \\ -n^2x + n & , \quad 0 < x \le \dfrac{1}{n} \\ 0 & , \quad \dfrac{1}{n} < x \le 1 \end{cases}.$$

Does your comparison allow you to conclude anything concerning the uniform convergence of the given series n $[0,1]$? Briefly justify your response.

14. For each $n \in \mathbb{J}$, let $f_n(x) = \begin{cases} 0 & , \text{if} \quad -1 \le x \le -\frac{1}{n} \\ \frac{nx+1}{2} & , \text{if} \quad -\frac{1}{n} < x < \frac{1}{n} \\ 0 & , \text{if} \quad \frac{1}{n} \le x \le 1 \end{cases}$.
Then $\{f_n\}_{n=1}^{\infty} \subset \mathcal{C}([-1,1])$ where $\mathcal{C}([-1,1])$ is the set of real-valued functions that are continuous on $[-1,1]$. Make use of $\{f_n\}_{n=1}^{\infty}$ to justify that the metric space $(\mathcal{C}([-1,1]), \rho)$ is not complete, where
$$\rho(f,g) = \int_{-1}^{1} |f(x) - g(x)|\, dx.$$

15. For each of the following families $\mathcal{F}$ of real-valued functions on the specified sets $\mathbf{\Omega}$, determine whether of not $\mathcal{F}$ is pointwise bounded, locally uniformly bounded, and/or uniformly bounded on $\mathbf{\Omega}$. Justify your conclusions.
    (a) $\mathcal{F} = \left\{1 - \frac{1}{nx} : n \in \mathbb{J}\right\}$, $\mathbf{\Omega} = (0,1]$
    (b) $\mathcal{F} = \left\{\frac{\sin nx}{\sqrt{n}} : n \in \mathbb{J}\right\}$, $\mathbf{\Omega} = [0,1]$
    (c) $\mathcal{F} = \left\{\frac{nx}{1+n^2x^2} : n \in \mathbb{J}\right\}$, $\mathbf{\Omega} = \mathbb{R}$
    (d) $\mathcal{F} = \left\{\frac{x^{2n}}{1+x^{2n}} : n \in \mathbb{J}\right\}$, $\mathbf{\Omega} = \mathbb{R}$
    (e) $\mathcal{F} = \left\{n^2x^n(1-x) : n \in \mathbb{J}\right\}$, $\mathbf{\Omega} = [0,1)$

16. Suppose that $\mathcal{F}$ is a family of real-valued functions on $\mathbb{R}$ that are differentiable on the interval $[a,b]$ and $\mathcal{F}' = \left\{f' : f \in \mathcal{F}\right\}$ is uniformly bounded on $[a,b]$. Prove that $\mathcal{F}$ is equicontinuous on (a,b).

17. Is $\mathcal{F} = \left\{nxe^{-nx^2} : n \in \mathbb{J} \wedge x \in \mathbb{R}\right\}$ uniformly bounded on $[0,\infty)$? State your position clearly and carefully justify it.

18. Is $\mathcal{G} = \left\{n \cos \frac{x}{2n} : n \in \mathbb{J} \wedge x \in \mathbb{R}\right\}$ equicontinuous on $\mathbb{R}$? State your position clearly and carefully justify it.

19. Is $\left\{ \sum_{k=1}^{n} \frac{k^2 x \sin kx}{1+k^4 x} \right\}_{n=1}^{\infty}$ uniformly convergent on $[0, \infty)$? State your position clearly and carefully justify it.

Chapter 9

Some Special Functions

Up to this point we have focused on the general properties that are associated with uniform convergence of sequences and series of functions. In this chapter, most of our attention will focus on series that are formed from sequences of functions that are polynomials having one and only one zero of increasing order. In a sense, these are series of functions that are "about as good as it gets." It would be even better if we were doing this discussion in the "Complex World"; however, we will restrict ourselves mostly to power series in the reals.

9.1 Power Series Over the Reals

In this section, we turn to series that are generated by sequences of functions $\left\{c_k (x-\alpha)^k\right\}_{k=0}^{\infty}$.

Definition 9.1.1 *A **power series in** $\mathbb{R}$ **about the point** $\alpha \in \mathbb{R}$ is a series in the form*

$$c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$$

where α and c_n, for $n \in \mathbb{J} \cup \{0\}$, are real constants.

Remark 9.1.2 *When we discuss power series, we are still interested in the different types of convergence that were discussed in the last chapter; namely, pointwise, uniform and absolute. In this context, for example, the power series $c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$ is said to be **pointwise convergent** on a set $S \subset \mathbb{R}$ if and only if, for each $x_0 \in S$, the series $c_0 + \sum_{n=1}^{\infty} c_n (x_0-\alpha)^n$ converges. If $c_0 + \sum_{n=1}^{\infty} c_n (x_0-\alpha)^n$*

is ***divergent****, then the power series* $c_0 + \sum_{n=1}^{\infty} c_n (x - \alpha)^n$ *is said to diverge at the point* x_0*.*

When a given power series $c_0 + \sum_{n=1}^{\infty} c_n (x - \alpha)^n$ is known to be pointwise convergent on a set $S \subset \mathbb{R}$, we define a function $f : S \longrightarrow \mathbb{R}$ by $f(x) = c_0 + \sum_{n=1}^{\infty} c_n (x - \alpha)^n$ whose range consists of the pointwise limits that are obtained from substituting the elements of S into the given power series.

We've already seen an example of a power series about which we know the convergence properties. The geometric series $1 + \sum_{n=1}^{\infty} x^n$ is a power series about the point 0 with coefficients $\{c_n\}_{n=0}^{\infty}$ satisfying $c_n = 1$ for all n. From the Convergence Properties of the Geometric Series and our work in the last chapter, we know that

- the series $\sum_{n=0}^{\infty} x^n$ is pointwise convergent to $\dfrac{1}{1-x}$ in $U = \{x \in \mathbb{R} : |x| < 1\}$,
- the series $\sum_{n=0}^{\infty} x^n$ is uniformly convergent in any compact subset of U, and
- the series $\sum_{n=0}^{\infty} x^n$ is not uniformly convergent in U.

We will see shortly that this list of properties is precisely the one that is associated with any power series on its segment (usually known as interval) of convergence. The next result, which follows directly from the Necessary Condition for Convergence, leads us to a characterization of the nature of the sets that serve as domains for convergence of power series.

Lemma 9.1.3 *If the series* $\sum_{n=0}^{\infty} c_n (x - \alpha)^n$ *converges for* $x_1 \neq \alpha$*, then the series converges absolutely for each* x *such that* $|x - \alpha| < |x_1 - \alpha|$*. Furthermore, there is a number* M *such that*

$$\left|c_n (x - \alpha)^n\right| \leq M \left(\frac{|x - \alpha|}{|x_1 - \alpha|}\right)^n \text{ for } |x - \alpha| \leq |x_1 - \alpha| \text{ and for all } n. \tag{9.1}$$

Proof. Suppose $\sum_{n=0}^{\infty} c_n (x - \alpha)^n$ converges at $x_1 \neq \alpha$. We know that a necessary condition for convergence is that the "nth terms" go to zero as n goes to infinity. Consequently, $\lim_{n \to \infty} c_n (x_1 - \alpha)^n = 0$ and, corresponding to $\varepsilon = 1$, there exists a positive integer K such that

$$n > K \Rightarrow \left|c_n (x_1 - \alpha)^n - 0\right| < 1.$$

Let $M = \max\left\{1, \max_{0 \le j \le K} c_j (x_1 - \alpha)^j\right\}$. Then

$$\left|c_n (x_1 - \alpha)^n\right| \le M \text{ for all } n \in \mathbb{J} \cup \{0\}.$$

For any fixed $x \in \mathbb{R}$ satisfying $|x - \alpha| \le |x_1 - \alpha|$, it follows that

$$\begin{aligned} \left|c_n (x - \alpha)^n\right| &= |c_n| \, |x - \alpha|^n = |c_n| \, |x_1 - \alpha|^n \left|\frac{x - \alpha}{x_1 - \alpha}\right|^n \\ &\le M \left|\frac{x - \alpha}{x_1 - \alpha}\right|^n \quad \text{for all } n \in \mathbb{J} \cup \{0\} \end{aligned}$$

as claimed in equation (9.1). Finally, for fixed $x \in \mathbb{R}$ satisfying $|x - \alpha| < |x_1 - \alpha|$, the Comparison Test yields the absolute convergence of $\sum_{n=0}^{\infty} c_n (x - \alpha)^n$. ■

The next theorem justifies that we have uniform convergence on compact subsets of a segment of convergence.

Theorem 9.1.4 *Suppose that the series $\sum_{n=0}^{\infty} c_n (x - \alpha)^n$ converges for $x_1 \neq \alpha$. Then the power series converges uniformly on $I = \{x \in \mathbb{R} : \alpha - h \le x \le \alpha + h\}$ for each nonnegative h such that $h < |x_1 - \alpha|$. Furthermore, there is a real number M such that*

$$\left|c_n (x - \alpha)^n\right| \le M \left(\frac{h}{|x_1 - \alpha|}\right)^n \text{ for } |x - \alpha| \le h < |x_1 - \alpha| \text{ and for all } n.$$

Proof. The existence of M such that $\left|c_n (x - \alpha)^n\right| \le M \left(\frac{|x-\alpha|}{|x_1-\alpha|}\right)^n$ was just shown in our proof of Lemma 9.1.3. For $|x - \alpha| \le h < |x_1 - \alpha|$, we have that

$$\frac{|x - \alpha|}{|x_1 - \alpha|} \le \frac{h}{|x_1 - \alpha|} < 1.$$

The uniform convergence now follows from the Weierstrass M-Test with $M_n = \left(\frac{h}{|x_1-\alpha|}\right)^n$. ■

Theorem 9.1.5 *For the power series $c_0 + \sum_{n=1}^{\infty} c_n (x - \alpha)^n$, either*

(i) the series converges only for $x = \alpha$; or

(ii) the series converges for all values of $x \in \mathbb{R}$; or

(iii) there is a positive real number R such that the series converges absolutely for each x satisfying $|x-\alpha| < R$, converges uniformly in $\{x \in \mathbb{R} : |x-\alpha| \leq R_0\}$ for any positive $R_0 < R$, and diverges for $x \in \mathbb{R}$ such that $|x-\alpha| > R$.

Proof. To see (i) and (ii), note that the power series $\sum_{n=1}^{\infty} n^n (x-\alpha)^n$ diverges for each $x \neq \alpha$, while $\sum_{n=0}^{\infty} \dfrac{(x-\alpha)^n}{n!}$ is convergent for each $x \in \mathbb{R}$. Now, for (iii), suppose that there is a real number $x_1 \neq \alpha$ for which the series converges and a real number x_2 for which it diverges. By Theorem 9.1.3, it follows that $|x_1-\alpha| \leq |x_2-\alpha|$. Let

$$S = \left\{ \rho \in \mathbb{R} : \sum_{n=0}^{\infty} \left| c_n (x-\alpha)^n \right| \text{ converges for } |x-\alpha| < \rho \right\}$$

and define

$$R = \sup\ S.$$

Suppose that x^* is such that $|x^*-\alpha| < R$. Then there exists a $\rho \in S$ such that $|x^*-\alpha| < \rho < R$. From the definition of S, we conclude that $\sum_{n=0}^{\infty} \left| c_n (x^*-\alpha)^n \right|$ converges. Since x^* was arbitrary, the given series is absolutely convergent for each x in $\{x \in \mathbb{R} : |x-\alpha| < R\}$. The uniform convergence in $\{x \in \mathbb{R} : |x-\alpha| \leq R_0\}$ for any positive $R_0 < R$ was justified in Theorem 9.1.4.

Next, suppose that $\hat{x} \in \mathbb{R}$ is such that $\left|\hat{x}-\alpha\right| = \hat{\rho} > R$. From Lemma 9.1.3, convergence of $\sum_{n=0}^{\infty} \left| c_n \left(\hat{x}-\alpha\right)^n \right|$ would yield absolute convergence of the given series for all x satisfying $|x-\alpha| < \hat{\rho}$ and place $\hat{\rho}$ in S which would contradict the definition of R. We conclude that for all $x \in \mathbb{R}$, $|x-\alpha| > R$ implies that $\sum_{n=0}^{\infty} \left| c_n (x-\alpha)^n \right|$ as well as $\sum_{n=0}^{\infty} c_n (x-\alpha)^n$ diverge. ■

The nth Root Test provides us with a formula for finding the radius of convergence, R, that is described in Theorem 9.1.5.

Lemma 9.1.6 *For the power series $c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$, let $\rho = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ and*

$$R = \begin{cases} +\infty & , \textit{if} \quad \rho = 0 \\ \dfrac{1}{\rho} & , \textit{if} \quad 0 < \rho < \infty \\ 0 & , \textit{if} \quad \rho = +\infty \end{cases} . \tag{9.2}$$

Then $c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$ *converges absolutely for each* $x \in (\alpha - R, \alpha + R)$, *converges uniformly in* $\{x \in \mathbb{R} : |x-\alpha| \le R_0\}$ *for any positive* $R_0 < R$, *and diverges for* $x \in \mathbb{R}$ *such that* $|x-\alpha| > R$. *The number* R *is called the radius of convergence for the given power series and the segment* $(\alpha - R, \alpha + R)$ *is called the "interval of convergence."*

Proof. For any fixed x_0, we have that

$$\limsup_{n\to\infty} \sqrt[n]{\left|c_n (x_0-\alpha)^n\right|} = \limsup_{n\to\infty} \left(|x_0-\alpha| \sqrt[n]{|c_n|}\right) = |x_0-\alpha|\,\rho.$$

From the Root Test, the series $c_0 + \sum_{n=1}^{\infty} c_n (x_0-\alpha)^n$ converges absolutely whenever $|x_0-\alpha|\,\rho < 1$ and diverges when $|x_0-\alpha|\,\rho > 1$. We conclude that the radius of convergence justified in Theorem 9.1.5 is given by equation (9.2). ■

Example 9.1.7 *Consider* $\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{3^n}(x-2)^n$. *Because* $\limsup_{n\to\infty} \sqrt[n]{\frac{2\,(2^n)}{3^n}} = \lim_{n\to\infty}\left(\frac{2}{3}\right)\sqrt[n]{2} = \frac{2}{3}$, *from Lemma 9.1.6, it follows that the given power series has radius of convergence* $\frac{3}{2}$. *On the other hand, some basic algebraic manipulations yield more information. Namely,*

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{3^n}(x-2)^n = -2\sum_{n=0}^{\infty}\left[\frac{(-2)}{3}(x-2)\right]^n = -2\frac{1}{1-\left[\frac{(-2)}{3}(x-2)\right]}$$

as long as $\left|\frac{(-2)}{3}(x-2)\right| < 1$, *from the Geometric Series Expansion Theorem. Therefore, for each* $x \in \mathbb{R}$ *such that* $|x-2| < \frac{3}{2}$, *we have that*

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{3^n}(x-2)^n = \frac{6}{1-2x}.$$

Another useful means of finding the radius of convergence of a power series follows from the Ratio Test when the limit of the exists.

Lemma 9.1.8 *Let* α *be a real constant and suppose that, for the sequence of nonzero real constants* $\{c_n\}_{n=0}^{\infty}$, $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$ *for* $0 \le L \le \infty$.

(i) *If* $L = 0$, *then* $c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$ *is absolutely convergent for all* $x \in \mathbb{R}$ *and uniformly convergent on compact subsets of* $\mathbb{R}$*;*

(ii) *If* $0 < L < \infty$, *then* $c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$ *is absolutely convergent* $\left(\alpha - \frac{1}{L}, \alpha + \frac{1}{L}\right)$, *uniformly convergent in any compact subset of* $\left(\alpha - \frac{1}{L}, \alpha + \frac{1}{L}\right)$, *and divergent for any* $x \in R$ *such that* $|x-\alpha| > \frac{1}{L}$*;*

(iii) *If* $L = \infty$, *then* $c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$ *is convergent only for* $x = \alpha$.

The proof is left as an exercise.

Remark 9.1.9 *In view of Lemma 9.1.8, whenever the sequence of nonzero real constants* $\{c_n\}_{n=0}^{\infty}$ *satisfies* $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$ *for* $0 \le L \le \infty$ *an alternative formula for the radius of convergence* R *of* $c_0 + \sum_{n=1}^{\infty} c_n (x-\alpha)^n$ *is given by*

$$R = \begin{cases} +\infty & , \text{if} \quad L = 0 \\ \dfrac{1}{L} & , \text{if} \quad 0 < L < \infty \\ 0 & , \text{if} \quad L = +\infty \end{cases} . \tag{9.3}$$

Example 9.1.10 *Consider* $\sum_{n=1}^{\infty} \frac{(-1)^n \, 2\cdot 4 \cdots (2n)}{1\cdot 4\cdot 7\cdots(3n-2)} (x+2)^n$.

Let $c_n = \frac{(-1)^n \, 2\cdot 4 \cdots (2n)}{1\cdot 4\cdot 7\cdots(3n-2)}$. *Then*

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{(-1)^n \, 2\cdot 4\cdots(2n)\cdot 2(n+1)}{1\cdot 4\cdot 7\cdots(3n-2)\cdot(3(n+1)-2)} \frac{1\cdot 4\cdot 7\cdots(3n-2)}{(-1)^n \, 2\cdot 4\cdots(2n)}\right| = \frac{2(n+1)}{3n+1} \longrightarrow \frac{2}{3}$$

as $n \to \infty$. *Consequently, from Lemma 9.1.8, the radius of convergence of the given power series is* $\frac{3}{2}$. *Therefore, the "interval of convergence" is* $\left(-\frac{8}{3}, -\frac{4}{3}\right)$.

The simple manipulations illustrated in Example 9.1.7 can also be used to derive power series expansions for rational functions.

Example 9.1.11 *Find a power series about the point* $\alpha = 1$ *that sums pointwise to* $\dfrac{8x-5}{(1+4x)(3-2x)}$ *and find its interval of convergence.*

Note that

$$\frac{8x-5}{(1+4x)(3-2x)} = \frac{1}{3-2x} + \frac{-2}{1+4x},$$

$$\frac{1}{3-2x} = \frac{1}{1-2(x-1)} = \sum_{n=0}^{\infty}[2(x-1)]^n = \sum_{n=0}^{\infty}2^n(x-1)^n \text{ for } |x-1| < \frac{1}{2}$$

and

$$\begin{aligned}\frac{-2}{1+4x} &= \frac{-2}{5}\frac{1}{1-\left[\left(\frac{-4}{5}\right)(x-1)\right]}\\ &= \frac{-2}{5}\sum_{n=0}^{\infty}\left[\left(\frac{-4}{5}\right)(x-1)\right]^n = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^{2n+1}}{5^{n+1}}(x-1)^n \text{ for } |x-1| < \frac{5}{4}.\end{aligned}$$

We have pointwise and absolute convergence of both sums for $|x-1| < \min\left\{\frac{1}{2}, \frac{5}{4}\right\}$. *It follows that*

$$\frac{8x-5}{(1+4x)(3-2x)} = \sum_{n=0}^{\infty}\left[(-1)^{n+1}\frac{2^{2n+1}}{5^{n+1}} + 2^n\right](x-1)^n \text{ for } |x-1| < \frac{1}{2}.$$

The nth partial sums of a power series are polynomials and polynomials are among the nicest functions that we know. The nature of the convergence of power series allows for transmission of the nice properties of polynomials to the limit functions.

Lemma 9.1.12 *Suppose that the series* $f(x) = \sum_{n=0}^{\infty} c_n(x-\alpha)^n$ *converges in* $\{x \in \mathbb{R} : |x-\alpha| < R\}$ *with* $R > 0$. *Then* f *is continuous and differentiable in* $(\alpha - R, \alpha + R)$, f' *is continuous in* $(\alpha - R, \alpha + R)$ *and*

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-\alpha)^{n-1} \text{ for } \alpha - R < x < \alpha + R.$$

Space for comments and scratch work.

Proof. For any x_1 such that $|x_1 - \alpha| < R$, there exists an $h \in \mathbb{R}$ with $0 < h < R$ such that $|x_1 - \alpha| < h$. Let $I = \{x : |x - \alpha| \leq h\}$. Then, by Theorem 9.1.4, $\sum_{n=0}^{\infty} c_n (x - \alpha)^n$ is uniformly convergent on I. From Theorem 8.3.3, f is continuous on I as the continuous limit of the polynomials $\sum_{j=0}^{n} c_j (x - \alpha)^j$. Consequently, f is continuous at x_1. Since the x_1 was arbitrary, we conclude that f is continuous in $|x - \alpha| < R$.

Note that $\sum_{n=1}^{\infty} nc_n (x - \alpha)^{n-1}$ is a power series whose limit, when it is convergent, is the limit of $\left\{s_n'\right\}$ where $s_n(x) = \sum_{j=0}^{n} c_j (x - \alpha)^j$. Thus, the second part of the theorem will follow from showing that $\sum_{n=1}^{\infty} nc_n (x - \alpha)^{n-1}$ converges at least where f is defined; i.e., in $|x - \alpha| < R$. Let $x_0 \in \{x \in \mathbb{R} : 0 < |x - \alpha| < R\}$. Then there exists an x^* with $|x_0 - \alpha| < |x^* - \alpha| < R$. In the proof of Lemma **??**, it was shown that there exists an $M > 0$ such that $\left|c_n (x^* - \alpha)^n\right| \leq M$ for $n \in \mathbb{J} \cup \{0\}$. Hence,

$$\left|nc_n (x_0 - \alpha)^{n-1}\right| = \frac{n}{|x^* - \alpha|} \cdot |c_n| \left|x^* - \alpha\right|^n \left|\frac{x_0 - \alpha}{x^* - \alpha}\right|^{n-1} \leq \frac{M}{|x^* - \alpha|} \cdot nr^{n-1}$$

for $r = \left|\dfrac{x_0 - \alpha}{x^* - \alpha}\right| < 1$. From the ratio test, the series $\sum_{n=1}^{\infty} nr^{n-1}$ converges. Thus, $\sum_{n=1}^{\infty} \dfrac{M}{|x^* - \alpha|} \cdot nr^{n-1}$ is convergent and we conclude that $\sum_{n=1}^{\infty} nc_n (x - \alpha)^{n-1}$ is convergent at x_0. Since x_0 was arbitrary we conclude that $\sum_{n=1}^{\infty} nc_n (x - \alpha)^{n-1}$ is convergent in $|x - \alpha| < R$. Applying the Theorems 9.1.4 and 8.3.3 as before leads to the desired conclusion for f'. ■

Theorem 9.1.13 (Differentiation and Integration of Power Series) *Suppose f is given by $\sum_{n=0}^{\infty} c_n (x - \alpha)^n$ for $x \in (\alpha - R, \alpha + R)$ with $R > 0$.*

(a) The function f possesses derivatives of all orders. For each positive integer m, the mth derivative is given by

$f^{(m)}(x) = \sum_{n=m}^{\infty} \binom{n}{m} c_n (x - \alpha)^{n-m}$ *for* $|x - \alpha| < R$

where $\binom{n}{m} = n(n-1)(n-2)\cdots(n-m+1)$.

(b) *For each x with $|x - \alpha| < R$, define the function F by $F(x) = \int_\alpha^x f(t)\,dt$. Then F is also given by $\sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-\alpha)^{n+1}$ which is obtained by term-by-term integration of the given series for f.*

(c) *The constants c_n are given by $c_n = \frac{f^{(n)}(\alpha)}{n!}$.*

Excursion 9.1.14 *Use the space that is provided to complete the following proof of the Theorem.*

Proof. Since (b) follows directly from Theorem 8.3.3 and (c) follows from substituting $x = \alpha$ in the formula from (a), we need only indicate some of the details for the proof of (a).

Let

$$S = \left\{ m \in \mathbb{N} : f^{(m)}(x) = \sum_{n=m}^{\infty} \binom{n}{m} c_n (x-\alpha)^{n-m} \ \ for \ \ |x-\alpha| < R \right\}$$

where $\binom{n}{m} = n(n-1)(n-2)\cdots(n-m+1)$. By Lemma 9.1.12, we know that $1 \in S$. Now suppose that $k \in S$ for some k; i.e.,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)\,c_n (x-\alpha)^{n-k} \text{ for } |x-\alpha| < R.$$

■

Remark 9.1.15 *Though we have restricted ourselves to power series in $\mathbb{R}$, note that none of what we have used relied on any properties of $\mathbb{R}$ that are not possessed* *by $\mathbb{C}$. With that in mind, we state the following theorem and note that the proofs are the same as the ones given above. However, the region of convergence is a disk rather than an interval.*

Theorem 9.1.16 *For the complex power series $c_0 + \sum_{n=1}^{\infty} c_n (z-\alpha)^n$ where α and c_n, for $n \in \mathbb{J} \cup \{0\}$, are complex constants, let $\rho = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ and*

$$R = \begin{cases} +\infty & , \text{if} \quad \rho = 0 \\ \dfrac{1}{\rho} & , \text{if} \quad 0 < \rho < \infty \\ 0 & , \text{if} \quad \rho = +\infty \end{cases} .$$

Then the series

(i) converges only for $z = \alpha$ when $R = 0$;

(ii) converges for all values of $z \in \mathbb{C}$ when $R = +\infty$; and

(iii) converges absolutely for each $z \in N_R(\alpha)$, converges uniformly in

$$\{x \in \mathbb{R} : |x-\alpha| \le R_0\} = \overline{N_{R_0}(\alpha)}$$

for any positive $R_0 < R$, and diverges for $z \in \mathbb{C}$ such that $|z-\alpha| > R$ whenever $0 < R < \infty$. In this case, R is called the radius of convergence for the series and $N_R(\alpha) = \{z \in \mathbb{C} : |z-\alpha| < R\}$ is the corresponding disk of convergence.

Both Lemma 9.1.12 and Theorem 9.1.13 hold for the complex series in their disks of convergence.

Remark 9.1.17 *Theorem 9.1.13 tells us that every function that is representable as a power series in some segment $(\alpha - R, \alpha + R)$ for $R > 0$ has continuous derivatives of all orders there and has the form $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n$. It is natural to ask if the converse is true? The answer to this question is no. Consider the function*

$$g(x) = \begin{cases} \exp\left(-1/x^2\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases} .$$

It follows from l'Hôpital's Rule that g is infinitely differentiable at $x = 0$ with $g^{(n)}(0) = 0$ for all $n \in \mathbb{J} \cup \{0\}$. Since the function is clearly not identically equal to zero in any segment about 0, we can't write g in the "desired form." This prompts us to take a different approach. Namely, we restrict ourselves to a class of functions that have the desired properties.

Definition 9.1.18 *A function that has continuous derivatives of all orders in the neighborhood of a point is said to be **infinitely differentiable at the point**.*

Definition 9.1.19 *Let f be a real-valued function on a segment I. The function f is said to be **analytic at the point** α if it is infinitely differentiable at $\alpha \in I$ and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n$ is valid in a segment $(\alpha - R, \alpha + R)$ for some $R > 0$. The function f is called **analytic on a set** if and only if it is analytic at each point of the set.*

Remark 9.1.20 *The example mentioned above tells us that infinitely differentiable at a point is not enough to give analyticity there.*

9.2 Some General Convergence Properties

There is a good reason why our discussion has said nothing about what happens at the points of closure of the segments of convergence. This is because there is no one conclusion that can be drawn. For example, each of the power series $\sum_{n=0}^{\infty} x^n$, $\sum_{n=0}^{\infty} \frac{x^n}{n}$, and $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ has the same "interval of convergence" $(-1, 1)$; however, the first is divergent at each of the endpoints, the second one is convergent at -1 and divergent at 1, and the last is convergent at both endpoints. The fine point to keep in mind is that the series when discussed from this viewpoint has nothing to do with the functions that the series represent if we stay in $(-1, 1)$. On the other hand, if a power series that represents a function in its segment is known to converge at an endpoint, we can say something about the relationship of that limit in relation to the given function. The precise set-up is given in the following result.

Theorem 9.2.1 *If $\sum_{n=0}^{\infty} c_n$ converges and $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $x \in (-1, 1)$, then $\lim_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} c_n$.*

Excursion 9.2.2 *Fill in what is missing in order to complete the following proof of Theorem 9.2.1.*

Proof. Let $s_n = \sum_{k=0}^{n} c_k$ and $s_{-1} = 0$. It follows that

$$\sum_{n=0}^{m} c_n x^n = \sum_{n=0}^{m} (s_n - s_{n-1}) x^n = \left((1-x) \sum_{n=0}^{m-1} s_n x^n \right) + s_m x^m .$$

Since $|x| < 1$ and $\lim_{m\to\infty} s_m = \sum_{n=0}^{\infty} c_n$, we have that $\lim_{m\to\infty} s_m x^m = 0$ and we conclude that

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = (1-x) \sum_{n=0}^{\infty} s_n x^n . \tag{9.4}$$

Let $s = \sum_{n=0}^{\infty} c_n$. For each $x \in (-1, 1)$, we know that $(1-x) \sum_{n=0}^{\infty} x^n = 1$. Thus,

$$s = (1-x) \sum_{n=0}^{\infty} s x^n . \tag{9.5}$$

Suppose that $\varepsilon > 0$ is given. Because $\lim_{n\to\infty} s_n = s$ there exists a positive integer M such that $\underset{(1)}{\underline{\hspace{3cm}}}$ implies that $|s_n - s| < \frac{\varepsilon}{2}$. Let

$$K = \max \left\{ \frac{1}{2}, \max_{0 \le j \le M} \left| s - s_j \right| \right\}$$

and

$$\delta = \begin{cases} \dfrac{1}{4} & \text{, if } \quad \varepsilon \ge 2KM \\ \\ \dfrac{\varepsilon}{2KM} & \text{, if } \quad \varepsilon < 2KM \end{cases} .$$

Note that, if $2KM \le \varepsilon$, then $\frac{KM}{4} = \frac{2KM}{8} \le \frac{\varepsilon}{8} < \frac{\varepsilon}{2}$. For $1 - \delta < x < 1$, it follows that

$$(1-x) \sum_{n=0}^{M} |s_n - s| \, |x|^n \le (1-x) \underset{(2)}{\underline{\hspace{1cm}}} \sum_{n=0}^{M} |x|^n < (1-x) \underset{(2)}{\underline{\hspace{1cm}}} M < \frac{\varepsilon}{2} . \tag{9.6}$$

Use equations (9.4) and (9.5), to show that, if $1-\delta < x < 1$, then

$$|f(x) - s| < \varepsilon.$$

(3)

■

Acceptable responses are: (1) $n > M$, (2) K, (3) Hopefully, you noted that $|f(x) - s|$ is bounded above by the sum of $(1-x)\sum_{n=0}^{M} |s_n - s|\,|x|^n$ and $(1-x)\sum_{n=M+1}^{\infty} |s_n - s|\,|x|^n$. The first summation is bounded above by $\frac{\varepsilon}{2}$ as shown in equation (9.6) while the latter summation is bounded above by $\frac{\varepsilon}{2}\left((1-x)\sum_{n=M+1}^{\infty} |x|^n\right)$; with $x > 0$ this yields that $(1-x)\sum_{n=M+1}^{\infty} |x|^n = (1-x)\sum_{n=M+1}^{\infty} x^n < (1-x)\sum_{n=0}^{\infty} |x|^n = 1$.

An application of Theorem 9.2.1 leads to a different proof of the following result concerning the Cauchy product of convergent numerical series.

Corollary 9.2.3 *If* $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, *and* $\sum_{n=0}^{\infty} c_n$ *are convergent to* A, B, *and* C, *respectively, and* $\sum_{n=0}^{\infty} c_n$ *is the Cauchy product of* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$, *then* $C = AB$.

Proof. For $0 \le x \le 1$, let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,\ g(x) = \sum_{n=0}^{\infty} b_n x^n, \text{ and } h(x) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$. Because each series converges absolutely for $|x| < 1$, for each fixed $x \in [0,1)$ we have that

$$f(x)\,g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n = h(x).$$

From Theorem 9.2.1,

$$\lim_{x\to 1^-} f(x) = A,\ \lim_{x\to 1^-} g(x) = B, \text{ and } \lim_{x\to 1^-} h(x) = C.$$

The result follows from the properties of limits. ■

One nice argument justifying that a power series is analytic at each point in its interval of convergence involves rearrangement of the power series. We will make use of the Binomial Theorem and the following result that justifies the needed rearrangement.

Lemma 9.2.4 *Given the double sequence* $\{a_{ij}\}_{i,j\in J}$ *suppose that* $\sum_{j=1}^{\infty} |a_{ij}| = b_i$ *and* $\sum_{i=1}^{\infty} b_i$ *converges. Then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Proof. Let $E = \{x_n : n \in \mathbb{J} \cup \{0\}\}$ be a denumerable set such that $\lim_{n\to\infty} x_n = x_0$ and, for each $i, n \in \mathbb{J}$ let

$$f_i(x_0) = \sum_{j=1}^{\infty} a_{ij} \text{ and } f_i(x_n) = \sum_{j=1}^{n} a_{ij}.$$

Furthermore, for each $x \in E$, define the function g on E by

$$g(x) = \sum_{i=1}^{\infty} f_i(x).$$

From the hypotheses, for each $i \in \mathbb{J}$, $\lim_{n\to\infty} f_i(x_n) = f_i(x_0)$. Furthermore, the definition of E ensures that for any sequence $\{w_k\}_{k=1}^{\infty} \subset E$ such that $\lim_{k\to\infty} w_k = x_0$, $\lim_{k\to\infty} f_i(w_k) = f_i(x_0)$. Consequently, from the Limits of Sequences Characterization for Continuity Theorem, for each $i \in \mathbb{J}$, f_i is continuous at x_0. Because $(\forall x)(\forall i)(i \in \mathbb{J} \wedge x \in E \Rightarrow |f_i(x)| \le b_i)$ and $\sum_{i=1}^{\infty} b_i$ converges, $\sum_{i=1}^{\infty} f_i(x)$ is uniformly convergent in E. From the Uniform Limit of Continuous Functions Theorem (8.3.3), g is continuous at x_0. Therefore,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} f_i(x_0) = g(x_0) = \lim_{n\to\infty} g(x_n).$$

Now

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \lim_{n\to\infty}\sum_{i=1}^{\infty} f_i\left(x_n\right) = \lim_{n\to\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{n} a_{ij}$$

$$= \lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{\infty} a_{ij} = \sum_{ji=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

■

Theorem 9.2.5 *Suppose that* $f\left(x\right) = \sum_{n=0}^{\infty} c_n x^n$ *converges in* $|x| < R$. *For* $a \in \left(-R, R\right)$, f *can be expanded in a power series about the point* $x = a$ *which converges in* $\{x \in \mathbb{R} : |x-a| < R - |a|\}$ *and* $f\left(x\right) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}\left(a\right)}{n!}\left(x-a\right)^n$.

In the following proof, extra space is provided in order to allow more room for scratch work to check some of the claims.

Proof. For $f\left(x\right) = \sum_{n=0}^{\infty} c_n x^n$ in $|x| < R$, let $a \in \left(-R, R\right)$. Then $f\left(x\right) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n\left[\left(x-a\right)+a\right]^n$and, from the Binomial Theorem,

$$f\left(x\right) = \sum_{n=0}^{\infty} c_n \sum_{j=0}^{n}\binom{n}{j} a^j \left(x-a\right)^{n-j} = \sum_{n=0}^{\infty}\sum_{j=0}^{n} c_n \binom{n}{j} a^j \left(x-a\right)^{n-j}.$$

We can think of this form of summation as a "summing by rows." In this context, the first row would could be written as $c_0\left(x-a\right)^0$, while the second row could be written as $c_1\left[\binom{1}{0}a^0\left(x-a\right)^1 + \binom{1}{1}a^1\left(x-a\right)^0\right]$. In general, the $\left(\ell+1\right)st$ row is given by

$$c_\ell\left[\sum_{j=0}^{\ell}\binom{\ell}{j}a^j\left(x-a\right)^{\ell-j}\right]$$

$$= c_\ell\left[\binom{\ell}{0}a^0\left(x-a\right)^\ell + \binom{\ell}{1}a^1\left(x-a\right)^{\ell-1} + \cdots + \binom{\ell}{\ell}a^\ell\left(x-a\right)^0\right].$$

In the space provided write 4-5 of the rows aligned in such a way as to help you envision what would happen if we decided to arrange the summation "by columns."

If $w_{nk} \underset{def}{=} \begin{cases} c_n\binom{n}{k}a^k\,(x-a)^{n-k} & \text{, if} \quad k \leq n \\ 0 & \text{, if} \quad k > n \end{cases}$, then it follows that

$$f\,(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n\,[(x-a)+a]^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} w_{nk}.$$

In view of Lemma 9.2.4, $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} w_{nk}\right) = \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} w_{nk}\right)$ whenever

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n} |c_n| \binom{n}{j} |a|^j\,|x-a|^{n-j} = \sum_{n=0}^{\infty} |c_n|\,(|x-a|+|a|)^n < \infty;$$

i.e., at least when $(|x-a|+|a|) < R$. Viewing the rearrangement as "summing by columns," yields that first column as $(x-a)^0\left[c_0a^0 + c_1a^1 + \cdots + \binom{n}{n}c_na^n + \cdots\right]$ and the second column as $(x-a)^1\left[\binom{1}{0}c_1a^0 + \binom{2}{1}c_2a^1 + \cdots + \binom{n}{n-1}c_na^{n-1} + \cdots\right]$. In general, we have that the $(k+1)\,st$ column if given by

$$(x-a)^k\left[c_ka^0 + \binom{k+1}{1}c_{k+1}a^1 + \cdots + \binom{n}{n-k}c_na^{n-k} + \cdots.\right]$$

Use the space that is provided to convince yourself concerning the form of the general term.

Hence, for any $x \in \mathbb{R}$ such that $|x-a| < R-|a|$, we have that

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty}(x-a)^k\left(\sum_{n=k}^{\infty}\binom{n}{n-k}c_n a^{n-k}\right) \\ &= \sum_{k=0}^{\infty}(x-a)^k\left(\sum_{n=k}^{\infty}\frac{n!}{k!\,(n-k)!}c_n a^{n-k}\right) \\ &= \sum_{k=0}^{\infty}(x-a)^k\,\frac{1}{k!}\left(\sum_{n=k}^{\infty}n\,(n-1)\,(n-2)\cdots(n-k+1)\,a^{n-k}c_n\right) \\ &= \sum_{k=0}^{\infty}(x-a)^k\,\frac{f^{(k)}(a)}{k!} \end{aligned}$$

as needed. ■

Theorem 9.2.6 (Identity Theorem) *Suppose that the series* $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *both converge in the segment* $S=(-R,R)$. *If*

$$E=\left\{x\in S:\sum_{n=0}^{\infty}a_n x^n=\sum_{n=0}^{\infty}b_n x^n\right\}$$

has a limit point in S, *then* $(\forall n)\,(n\in\mathbb{J}\cup\{0\}\Rightarrow a_n=b_n)$ *and* $E=S$.

Excursion 9.2.7 *Fill in what is missing in order to complete the following proof of the Identity Theorem.*

Proof. Suppose that the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ both converge in the segment $S = (-R, R)$ and that

$$E = \left\{ x \in S : \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \right\}$$

has a limit point in S. For each $n \in \mathbb{J} \cup \{0\}$, let $c_n = a_n - b_n$. Then $f(x) \underset{def}{=} \sum_{n=0}^{\infty} c_n x^n = 0$ for each $x \in E$. Let

$$A = \left\{ x \in S : x \in E' \right\} \text{ and } B = S - A = \{ x \in S : x \notin A \}$$

where E' denotes the set of limit points of E. Note that S is a connected set such that $S = A \cup B$ and $A \cap B = \emptyset$. First we will justify that B is open. If B is empty, then we are done. If B is not empty and not open, then there exists a $w \in B$ such that $\neg (\exists N_\delta(w)) (N_\delta(w) \subset B)$.

(1)

Next we will show that A is open. Suppose that $x_0 \in A$. Because $x_0 \in S$, by Theorem 9.2.5,

$$f(x) = \sum_{n=0}^{\infty} d_n (x - x_0)^n \text{ for } \underset{(2)}{\underline{\hspace{3cm}}}.$$

Suppose that $T = \left\{ j \in \mathbb{J} \cup \{0\} : d_j \neq 0 \right\} \neq \emptyset$. By the $\underset{(3)}{\underline{\hspace{3cm}}}$, T has a least element, say k. It follows that we can write $f(x) = (x - x_0)^k g(x)$ where $g(x) = \sum_{m=0}^{\infty} d_{k+m} (x - x_0)^n$ for $\underset{(2)}{\underline{\hspace{3cm}}}$. Because g is continuous at x_0, we know that $\lim_{x \to x_0} g(x) = \underset{(4)}{\underline{\hspace{1.5cm}}} = \underset{(5)}{\underline{\hspace{1.5cm}}} \neq 0$. Now we will make use of the fact that $\dfrac{|g(x_0)|}{2} > 0$ to show that there exists $\delta > 0$ such that $g(x) \neq 0$ for $|x - x_0| < \delta$.

(6)

Hence, $g(x) \neq 0$ for $|x - x_0| < \delta$ from which it follows that

$$f(x) = (x - x_0)^k g(x) \neq 0$$

in $\underset{(7)}{\underline{\qquad\qquad\qquad}}$. But this contradicts the claim that x_0 is a limit point of zeroes of f. Therefore, $\underset{(8)}{\underline{\qquad\qquad\qquad}}$ and we conclude that $\underset{(9)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}$. Thus, $f(x) = \sum_{n=0}^{\infty} d_n (x - x_0)^n = 0$ for all x in a neighborhood $N(x_0)$ of x_0. Hence, $N(x_0) \subset A$. Since x_0 was arbitrary, we conclude that

$$(\forall w)\left(w \in A \Rightarrow \underset{(10)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}} \right); \text{ i.e.,}$$

$\underset{(11)}{\underline{\qquad\qquad\qquad}}$.

Because S is a connected set for which A and B are open sets such that $S = A \cup B$, $A \neq \emptyset$, and $A \cap B = \emptyset$, we conclude that $\underset{(12)}{\underline{\qquad\qquad\qquad}}$. ■

Acceptable responses are: (1) Your argument should have generated a sequence of elements of E that converges to w. This necessitated an intermediate step because at each step you could only claim to have a point that was in E'. For example, if $N_\delta(w)$ is not contained in B, then there exists a $v \in S$ such that $v \notin B$ which places v in E'. While this does not place v in E, it does insure that any neighborhood of v contains an element of E. Let u_1 be an element of E such that $u_1 \neq w$ and $|u_1 - w| < \delta$. The process can be continued to generate a sequence of elements of E, $\{u_n\}_{n=1}^{\infty}$, that converges to w. This would place w in $A \cap B$ which contradicts the choice of B. (2) $|x - x_0| < R - |x_0|$, (3) Well-Ordering Principle, (4) $g(x_0)$, (5) d_k, (6) We've seen this one a few times before. Corresponding to $\varepsilon = \dfrac{|g(x_0)|}{2}$, there exists a $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon$. The (other) triangular inequality, then yields that $|g(x_0)| - |g(x)| < \dfrac{|g(x_0)|}{2}$ which implies that $|g(x)| > \dfrac{|g(x_0)|}{2}$ whenever $|x - x_0| < \delta$. (7) $0 < |x - x_0| < \delta$, (8) $T = \emptyset$, (9) $(\forall n)(n \in \mathbb{J} \cup \{0\} \Rightarrow d_n = 0)$, (10) $(\exists N(w))(N(w) \subset A)$, (11) A is open, (12) B is empty.

9.3 Designer Series

With this section, we focus attention on one specific power series expansion that satisfies some special function behavior. Thus far we have been using the definition of e that is developed in most elementary calculus courses, namely, $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. There are alternative approaches that lead us to e. In this section, we will obtain e as the value of power series at a point. In Chapter 3 of Rudin, e was defined as $\sum_{n=0}^{\infty}\frac{1}{n!}$ and it was shown that $\sum_{n=0}^{\infty}\frac{1}{n!} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. We get to this point from work on a specially chosen power series. The series leads to a definition for the function e^x and $\ln x$ as well as a "from series perspective" view of trigonometric functions.

For each $n \in \mathbb{J}$, if $c_n = (n!)^{-1}$, then $\limsup_{n\to\infty}\left(|c_{n+1}|\,|c_n|^{-1}\right) = 0$. Hence, the Ratio Test yields that $\sum_{n=0}^{\infty} c_n z^n$ is absolutely convergent for each $z \in \mathbb{C}$. Consequently, we can let

$$E(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!} \text{ for z} \in \mathbb{C}. \tag{9.7}$$

Complete the following exercises in order to obtain some general properties of $E(z)$. If you get stuck, note that the following is a working excursion version of a subset of what is done on pages 178-180 of our text.

From the absolute convergence of the power series given in (9.7), for any fixed $z, w \in \mathbb{C}$, the Cauchy product, as defined in Chapter 4, of $E(z)$ and $E(w)$ can be written as

$$E(z)\,E(w) = \left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{w^n}{n!}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{z^k w^{n-k}}{k!\,(n-k)!}.$$

From $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$, it follows that

$$\begin{aligned} E(z)\,E(w) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{n!}\frac{n!}{k!\,(n-k)!}z^k w^{n-k} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{k=0}^{n}\binom{n}{k}z^k w^{n-k}\right) \\ &= \sum_{n=0}^{\infty}\frac{(z+w)^n}{n!}. \end{aligned}$$

Therefore,

$$E(z)\,E(w) = E(z+w)\,. \tag{9.8}$$

Suppose there exists a $\zeta \in \mathbb{C}$ such that $E(\zeta) = 0$. Taking $z = \zeta$ and $w = -\zeta$ in (9.8) yields that

$$E(\zeta)\,E(-\zeta) = E(0) = 1 \tag{9.9}$$

which would contradict our second Property of the Additive Identity of a Field (Proposition 1.1.4) from which we have to have that $E(\zeta)\,E(w) = 0$ for all $w \in \mathbb{C}$. Consequently $(\forall z)\,(z \in \mathbb{C} \Rightarrow E(z) \neq 0)$.

1. For x real, use basic bounding arguments and field properties to justify each of the following.

    (a) $(\forall x)\,(x \in \mathbb{R} \Rightarrow E(x) > 0)$

    (b) $\lim\limits_{x \to -\infty} E(x) = 0$

    (c) $(\forall x)\,(\forall y)\,[(x, y \in \mathbb{R} \wedge 0 < x < y) \Rightarrow (E(x) < E(y) \wedge E(-y) < E(-x))]$

What you have just shown justifies that $E(x)$ over the reals is a strictly increasing function that is positive for each $x \in \mathbb{R}$.

2. Use the definition of the derivative to prove that

$$(\forall z)\left(z \in \mathbb{C} \Rightarrow E'(z) = E(z)\right).$$

Note that when x is real, $E'(x) = E(x)$ and $(\forall x)(x \in \mathbb{R} \Rightarrow E(x) > 0)$ with the Monotonicity Test yields an alternative justification that E is increasing in $\mathbb{R}$.

A straight induction argument allows us to claim from (9.8) that

$$(\forall n)\left[n \in \mathbb{J} \Rightarrow E\left(\sum_{j=1}^{n} z_j\right) = \prod_{j=1}^{n} E\left(z_j\right)\right]. \tag{9.10}$$

3. Complete the justification that

$$E(1) = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

For each $n \in \mathbb{J}$, let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} \qquad \textit{and} \qquad t_n = \left(1 + \frac{1}{n}\right)^n$$

(a) Use the Binomial Theorem to justify that,

$$\begin{aligned} t_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots \\ + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right). \end{aligned}$$

(b) Use part (a) to justify that $\limsup_{n\to\infty} t_n \leq E(1)$.

(c) For $n > m > 2$, justify that

$$\begin{aligned} t_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots \\ + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right). \end{aligned}$$

(d) Use the inequality you obtain by keeping m fixed and letting $n \to \infty$ in the equation from part (c) to obtain a lower bound on $\liminf_{n\to\infty} t_n$ and an upper bound on s_m for each m.

(e) Finish the argument.

4. Use properties of E to justify each of the following claims.

(a) $(\forall n)\,(n \in \mathbb{J} \Rightarrow E(n) = e^n)$.

(b) $(\forall u)\,(u \in \mathbb{Q} \wedge u > 0 \Rightarrow E(u) = e^u)$

Using field properties and the density of the rationals can get us to a justification that $E(x) = e^x$ for x real.

5. Show that, for $x > 0$, $e^x > \dfrac{x^{n+1}}{(n+1)!}$ and use the inequality to justify that $\lim\limits_{x \to +\infty} x^n e^{-x} = 0$ for each $n \in \mathbb{J}$.

9.3.1 Another Visit With the Logarithm Function

Because the function $E \restriction_{\mathbb{R}}$ is strictly increasing and differentiable from $\mathbb{R}$ into $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, by the Inverse Function Theorem, $E \restriction_{\mathbb{R}}$ has an inverse function $L : \mathbb{R}^+ \to \mathbb{R}$, defined by $E(L(y)) = y$ that is strictly increasing and differentiable on $\mathbb{R}^+$. For $x \in \mathbb{R}$, we have that $L(E(x)) = x$, for x real and the Inverse Differentiation Theorem yields that

$$L'(y) = \frac{1}{y} \text{ for } y > 0 \tag{9.11}$$

where $y = E(x)$. Since $E(0) = 1$, $L(1) = 0$ and (9.11) implies that

$$L(y) = \int_1^y \frac{dx}{x}$$

which gets us back to the natural logarithm as it was defined in Chapter 7 of these notes. A discussion of some of the properties of the natural logarithm is offered on pages 180-182 of our text.

9.3.2 A Series Development of Two Trigonometric Functions

The development of the real exponential and logarithm functions followed from restricting consideration of the complex series $E(z)$ to $\mathbb{R}$. In this section, we consider $E(z)$ restricted the subset of $\mathbb{C}$ consisting of numbers that are purely imaginary. For $x \in \mathbb{R}$,

$$E(ix) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i)^n x^n}{n!}.$$

Since

$$i^n = \begin{cases} 1 & \text{, if } \ 4 \mid n \\ i & \text{, if } \ 4 \mid (n-1) \\ -1 & \text{, if } \ 4 \mid (n-2) \\ -i & \text{, if } \ 4 \mid (n-3) \end{cases} \quad \textit{and} \quad (-i)^n = \begin{cases} 1 & \text{, if } \ 4 \mid n \\ -i & \text{, if } \ 4 \mid (n-1) \\ -1 & \text{, if } \ 4 \mid (n-2) \\ i & \text{, if } \ 4 \mid (n-3) \end{cases},$$

it follows that each of

$$C(x) = \frac{1}{2}\left[E(ix) + E(-ix)\right] \quad \textit{and} \quad S(x) = \frac{1}{2i}\left[E(ix) - E(-ix)\right] \tag{9.12}$$

have real coefficients and are, thus, real valued functions. We also note that

$$E(ix) = C(x) + iS(x) \tag{9.13}$$

from which we conclude that $C(x)$ and $S(x)$ are the real and imaginary parts of $E(ix)$, for $x \in \mathbb{R}$.

Complete the following exercises in order to obtain some general properties of $C(x)$ and $S(x)$ for $x \in \mathbb{R}$. If you get stuck, note that the following is a working excursion version of a subset of what is done on pages 182-184 of our text. Once completed, the list of properties justify that $C(x)$ and $S(x)$ for $x \in \mathbb{R}$ correspond to the $\cos x$ and $\sin x$, respectively, though appeal to triangles or the normal geometric view is never made in the development.

1. Show that $|E(ix)| = 1$.

2. By inspection, we see that $C(0) = 1$ and $S(0) = 0$. Justify that $C'(x) = -S(x)$ and $S'(x) = C(x)$.

3. Prove that $(\exists x)\left(x \in \mathbb{R}^+ \wedge C(x) = 0\right)$.

4. Justify that there exists a smallest positive real number x_0 such that $C(x_0) = 0$.

5. Define the symbol π by $\pi = 2x_0$ where x_0 is the number from #4 and justify each of the following claims.

(a) $S\left(\frac{\pi}{2}\right) = 1$

(b) $E\left(\frac{\pi i}{2}\right) = i$

(c) $E(\pi i) = -1$

(d) $E(2\pi i) = 1$

It follows immediately from equation (9.8) that E is periodic with period $2\pi i$; i.e.,

$$(\forall z)\,(z \in \mathbb{C} \Rightarrow E(z + 2\pi i) = E(z))\,.$$

Then the formulas given in equation (9.12) immediately yield that both C and S are periodic with period $2\pi i$.

Also shown in Theorem 8.7 of our text is that $(\forall t)\,(t \in (0, 2\pi) \Rightarrow E(it) \neq 1)$ and

$$(\forall z)\,[(z \in \mathbb{C} \wedge |z| = 1) \Rightarrow (\exists! t)\,(t \in [0, 2\pi) \wedge E(it) = z)]\,.$$

The following space is provided for you to enter some helpful notes towards justifying each of these claims.

9.4 Series from Taylor's Theorem

The following theorem supplies us with a sufficient condition for a given function to be representable as a power series. The statement and proof should be strongly reminiscent of Taylor's Approximating Polynomials Theorem that we saw in Chapter 6.

Theorem 9.4.1 (Taylor's Theorem with Remainder) *For* $a < b$, *let* $I = [a, b]$. *Suppose that* f *and* $f^{(j)}$ *are in* $\mathcal{C}(I)$ *for* $1 \leq j \leq n$ *and that* $f^{(n+1)}$ *is defined for each* $x \in \operatorname{Int}(I)$. *Then, for each* $x \in I$, *there exists a* ξ *with* $a < \xi < x$ *such that*

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j + R_n(x)$$

where $R_n(x) = \dfrac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}$ *is known as the Lagrange Form of the Remainder* .

Excursion 9.4.2 *Fill in what is missing to complete the following proof.*

Proof. It suffices to prove the theorem for the case $x = b$. Since f and $f^{(j)}$ are in $\mathcal{C}(I)$ for $1 \leq j \leq n$, $R_n = f(b) - \sum_{j=0}^{n} \dfrac{f^{(j)}(a)}{j!}(b-a)^j$ is well defined. In order to find a different form of R_n, we introduce a function φ. For $x \in I$, let

$$\varphi(x) = f(b) - \sum_{j=0}^{n} \frac{f^{(j)}(x)}{j!}(b-x)^j - \frac{(b-x)^{n+1}}{(b-a)^{n+1}} R_n.$$

From the hypotheses and the properties of continuous and $\underset{(1)}{\underline{\hspace{3cm}}}$ functions, we know that φ is $\underset{(2)}{\underline{\hspace{3cm}}}$ and differentiable for each $x \in I$. Furthermore,

$$\varphi(a) = \underset{(3)}{\underline{\hspace{9cm}}} = \underset{(4)}{\underline{\hspace{1cm}}}$$

and $\varphi(b) = 0$. By $\underset{(5)}{\rule{4cm}{0.4pt}}$, there exists a $\xi \in I$ such that $\varphi'(\xi) = 0$. Now

$$\varphi'(x) = -\sum_{j=1}^{n}\left[\frac{-f^{(j)}(x)}{(j-1)!}(b-x)^{j-1}\right] - \underset{(6)}{\rule{4cm}{0.4pt}} + \frac{(n+1)(b-x)^n}{(b-a)^{n+1}}R_n.$$

Because

$$\begin{aligned}\sum_{j=0}^{n}\frac{f^{(j+1)}(x)}{j!}(b-x)^j &= f'(x) + \sum_{j=1}^{n}\frac{f^{(j+1)}(x)}{j!}(b-x)^j \\ &= f'(x) + \sum_{j=2}^{n+1}\frac{f^{(j)}(x)}{(j-1)!}(b-x)^{j-1},\end{aligned}$$

it follows that

$$\begin{aligned}&\varphi'(x) - \tfrac{(n+1)(b-x)^n}{(b-a)^{n+1}}R_n \\ &= \textstyle\sum_{j=2}^{n}\frac{f^{(j)}(x)}{(j-1)!}(b-x)^{j-1} - \left(\sum_{j=2}^{n+1}\frac{f^{(j)}(x)}{(j-1)!}(b-x)^{j-1}\right) \\ &= \underset{(7)}{\rule{4cm}{0.4pt}}.\end{aligned}$$

If $\varphi'(\xi) = 0$, then $\dfrac{f^{(n+1)}(\xi)}{n!}(b-\xi)^n = \dfrac{(n+1)(b-\xi)^n}{(b-a)^{n+1}}R_n$. Therefore,

$\underset{(8)}{\rule{4cm}{0.4pt}}$. ■

***Acceptable responses are: (1) differentiable, (2) continuous, (3) $f(b) - \sum_{j=0}^{n}\frac{f^{(j)}(x)}{j!}(b-a)^j - R_n$, (4) 0, (5) Rolle's or the Mean-Value Theorem, (6) $\sum_{j=0}^{n}\dfrac{f^{(j+1)}(x)}{j!}(b-x)^j$, (7) $-\dfrac{f^{(n+1)}(x)}{n!}(b-x)^n$,
(8) $R_n = \dfrac{f^{(n+1)}(\xi)(b-a)^{n+1}}{(n+1)!}$.***

Remark 9.4.3 *Notice that the inequality* $a < b$ *was only a convenience for framing the argument; i.e., if we have the conditions holding in a neighborhood of a point* α *we have the Taylor's Series expansion to the left of* α *and to the right of* α. *In this case, we refer to the expansion as a Taylor's Series with Lagrange Form of the Remainder about* α.

Corollary 9.4.4 *For* $\alpha \in \mathbb{R}$ *and* $R > 0$, *suppose that* f *and* $f^{(j)}$ *are in* $\mathcal{C}((\alpha - R, \alpha + R))$ *for* $1 \leq j \leq n$ *and that* $f^{(n+1)}$ *is defined for each* $x \in (\alpha - R, \alpha + R)$. *Then, for each* $x \in (\alpha - R, \alpha + R)$, *there exists a* $\xi \in (\alpha - R, \alpha + R)$ *such that*

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(\alpha)}{j!}(x-\alpha)^j + R_n$$

where $R_n = \dfrac{f^{(n+1)}(\xi)(x-\alpha)^{n+1}}{(n+1)!}$.

9.4.1 Some Series To Know & Love

When all of the derivatives of a given function are continuous **in a neighborhood of a point** α, the Taylor series expansion about α simply takes the form $f(x) = \sum_{j=0}^{\infty} \dfrac{f^{(j)}(\alpha)}{j!}(x-\alpha)^j$ with its radius of convergence being determined by the behavior of the coefficients. Alternatively, we can justify the series expansion by proving that the remainder goes to 0 as $n \to \infty$. There are several series expansions that we should just know and/or be able to use.

Theorem 9.4.5

(a) For all real α and x, we have

$$e^x = e^{\alpha}\sum_{n=0}^{\infty}\frac{(x-\alpha)^n}{n!}. \tag{9.14}$$

(b) For all real α and x, we have

$$\sin x = \sum_{n=0}^{\infty}\frac{\sin\left(\alpha + \frac{n\pi}{2}\right)}{n!}(x-\alpha)^n \tag{9.15}$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{\cos\left(\alpha + \frac{n\pi}{2}\right)}{n!} (x - \alpha)^n . \tag{9.16}$$

(c) For $|x| < 1$, we have

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \tag{9.17}$$

and

$$\arctan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2n - 1}. \tag{9.18}$$

(d) *The Binomial Series Theorem.* For each $m \in \mathbb{R}^1$ and for $|x| < 1$, we have

$$(1 + x)^m = 1 + \sum_{n=1}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n. \tag{9.19}$$

We will offer proofs for (a), and the first parts of (b) and (c). A fairly complete sketch of a proof for the Binomial Series Theorem is given after discussion of a different form of Taylor's Theorem.

Proof. Let $f(x) = e^x$. Then f is continuously differentiable on all of $\mathbb{R}$ and $f^{(n)}(x) = e^x$ for each $n \in \mathbb{J}$. For $\alpha \in \mathbb{R}$, from Taylor's Theorem with Remainder, we have that

$$f(x) = e^x = e^{\alpha} \sum_{j=0}^{n} \frac{1}{j!} (x - \alpha)^j + R_n(\alpha, x) \text{ where } R_n = \frac{e^{\xi} (x - \alpha)^{n+1}}{(n + 1)!}$$

where ξ is between α and x. Note that

$$0 \leq \left| e^{\alpha} \sum_{n=0}^{\infty} \frac{(x - \alpha)^n}{n!} - f(x) \right| = |R_n| .$$

Furthermore, because $x > \alpha$ with $\alpha < \xi < x$ implies that $e^{\xi} < e^{x}$, while $x < a$ yields that $x < \xi < \alpha$ and $e^{\xi} < e^{\alpha}$,

$$|R_n| = \frac{e^{\xi}\,|x-\alpha|^{n+1}}{(n+1)!} \leq \begin{cases} e^{x}\dfrac{|x-\alpha|^{n+1}}{(n+1)!} & \text{, if} \quad x \geq \alpha \\ e^{\alpha}\dfrac{|x-\alpha|^{n+1}}{(n+1)!} & \text{, if} \quad x < \alpha \end{cases}.$$

Since $\lim_{n\to\infty} \frac{k^n}{n!} = 0$ for any fixed $k \in \mathbb{R}$, we conclude that $R_n \to 0$ as $n \to \infty$. From the Ratio Test, $\sum_{n=0}^{\infty} \frac{(x-\alpha)^n}{n!}$ is convergent for all $x \in \mathbb{R}$. We conclude that the series given in (9.14) converges to f for each x and α.

The expansion claimed in (9.17) follows from the Integrability of Series because

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} \qquad \textit{and} \qquad \frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n\, t^n \text{ for } |t| < 1.$$

■

There are many forms of the remainder for "Taylor expansions" that appear in the literature. Alternatives can offer different estimates for the error entailed when a Taylor polynomial is used to replace a function in some mathematical problem. The integral form is given with the following

Theorem 9.4.6 (Taylor's Theorem with Integral Form of the Remainder) *Suppose that* f *and its derivatives of order up to* $n+1$ *are continuous on a segment* I *containing* α. *Then, for each* $x \in I$, $f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(\alpha)(x-\alpha)^j}{j!} + R_n(\alpha, x)$ *where*

$$R_n(\alpha, x) = \int_{\alpha}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt.$$

Proof. Since f' is continuous on the interval I, we can integrate the derivative to obtain

$$f(x) = f(\alpha) + \int_{\alpha}^{x} f'(t)\, dt.$$

As an application of Integration-by-Parts, for fixed x, corresponding to $u = f'(t)$ and $dv = dt$, $du = f''(t)\,dt$ and we can choose $v = -(x-t)$. Then

$$\begin{aligned} f(x) &= f(\alpha) + \int_\alpha^x f'(t)\,dt = f(\alpha) - f'(t)(x-t)\,|_{t=\alpha}^{t=x} + \int_\alpha^x (x-t)\, f''(t)\,dt \\ &= f(\alpha) + f'(\alpha)(x-\alpha) + \int_\alpha^x (x-t)\, f''(t)\,dt. \end{aligned}$$

Next suppose that

$$f(x) = \sum_{n=0}^{k} \frac{f^{(j)}(\alpha)(x-\alpha)^j}{j!} + \int_\alpha^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\,dt$$

and $f^{(k+1)}$ is differentiable on I. Then Integration-by-Parts can be applied to $\int_\alpha^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\,dt$; taking $u = f^{(k+1)}(t)$ and $dv = \dfrac{(x-t)^k}{k!}dt$ leads to $u = f^{(k+2)}(t)\,dt$ and $v = -\dfrac{(x-t)^{k+1}}{(k+1)!}$. Substitution and simplification justifies the claim. ■

As an application of Taylor's Theorem with Integral Form of Remainder, complete the following proof of the *The Binomial Series Theorem.*

Proof. For fixed $m \in \mathbb{R}^1$ and $x \in \mathbb{R}$ such that $|x| < 1$, from Taylor's Theorem with Integral Form of Remainder, we have

$$(1+x)^m = 1 + \sum_{n=1}^{k} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n + R_k(0,x).$$

where

$$R_k(0,x) = \int_0^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\,dt.$$

We want to show that

$$R_k(0,x) = \int_0^x m(m-1)\cdots(m-k)\,\frac{(x-t)^k}{k!}(1+t)^{m-k-1}\,dt \longrightarrow 0 \text{ as } k \to \infty$$

for all x such that $|x| < 1$. Having two expressions in the integrand that involve a power k suggests a rearrangement of the integrand; i.e.,

$$R_k(0,x) = \int_0^x \frac{m(m-1)\cdots(m-k)}{k!}\left(\frac{x-t}{1+t}\right)^k (1+t)^{m-1}\,dt.$$

We discuss the behavior of $(1+t)^{m-1}$, when t is between 0 and x, and $\int_0^x \left(\frac{x-t}{1+t}\right)^k dt$ separately.

On one hand, we have that

$$(1+t)^{m-1} \le 1 \text{ whenever } (m \ge 1 \wedge -1 < t \le 0) \vee (m \le 1 \wedge 1 > t \ge 0).$$

On the other hand, because t is between 0 and x, if $m \ge 1 \wedge x \ge 0$ or $m \le 1 \wedge x \le 0$, then

$$g(t) = (1+t)^{m-1} \text{ implies that } g'(t) = (m-1)(1+t)^{m-2}\begin{cases} > 0 & \text{for} \quad m > 1 \\ \\ < 0 & \text{for} \quad m < 1 \end{cases}.$$

Consequently, if $m \ge 1 \wedge x \ge 0$, then $0 < t < x$ and g increasing yields the $g(t) \le g(x)$; while $m \le 1 \wedge x \le 0$, $0 < t < x$ and g decreasing, implies that $g(x) \ge g(t)$. With this in mind, define $C_m(x)$, for $|x| < 1$ by

$$C_m(x) = \begin{cases} (1+x)^{m-1} & , \quad m \ge 1, x \ge 0 \text{ OR } m \le 1, x \le 0 \\ 1 & , \quad m \ge 1, x < 0 \text{ OR } m \le 1, x \ge 0 \end{cases}.$$

We have shown that

$$(1+t)^{m-1} = C_m(t) \le C_m(x), \text{ for } t \text{ between 0 and } x. \tag{9.20}$$

Next, we turn to $\int_0^x \left(\frac{x-t}{1+t}\right)^k dt$. Since we want to bound the behavior in terms of x or a constant, we want to get the x out of the limits of integration. The standard way to do this is to effect a change of variable. Let $t = xs$. Then $dt = xds$ and

$$\int_0^x \left(\frac{x-t}{1+t}\right)^k dt = \int_0^1 x^{k+1}\left(\frac{1-s}{1+xs}\right)^k ds.$$

Since $s(1+x) \geq 0$, we immediately conclude that $\left(\frac{1-s}{1+xs}\right)^k \leq 1$. Hence, it follows that

$$\left|\int_0^x \left(\frac{x-t}{1+t}\right)^k dt\right| \leq |x|^{k+1}. \tag{9.21}$$

From (9.20) and (9.21), if follows that

$$\begin{aligned} 0 &\leq |R_k(0,x)| \\ &\leq \int_0^1 \frac{m(m-1)\cdots(m-k)}{k!}|x|^{k+1} C_m(x)\,dt \\ &= \frac{|m(m-1)\cdots(m-k)|}{k!}|x|^{k+1} C_m(x). \end{aligned}$$

For $u_k(x) = \dfrac{|m(m-1)\cdots(m-k)|}{k!}|x|^{k+1} C_m(x)$ consider $\sum_{n=1}^{\infty} u_n(x)$. Because

$$\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \left|\frac{m}{n+1} - 1\right| |x| \to |x| \text{ as } n \to \infty,$$

$\sum_{n=1}^{\infty} u_n(x)$ is convergent for $|x| < 1$. From the nth term test, it follows that $u_k(x) \to 0$ as $k \to \infty$ for all x such that $|x| < 1$. Finally, from the Squeeze Principle, we conclude that $R_k(0,x) \to 0$ as $k \to \infty$ for all x with $|x| < 1$. ■

9.4.2 Series From Other Series

There are some simple substitutions into power series that can facilitate the derivation of series expansions from some functions for which series expansions are "known." The proof of the following two examples are left as an exercise.

Theorem 9.4.7 *Suppose that* $f(u) = \sum_{n=0}^{\infty} c_n (u-b)^n$ *for* $|u-b| < R$ *with* $R > 0$.

(a) If $b = kc + d$ *with* $k \neq 0$, *then* $f(kx+d) = \sum_{n=0}^{\infty} c_n k^n (x-c)^n$ *for* $|x-c| < \dfrac{R}{|k|}$.

(b) *For every fixed positive integer k,* $f\left[(x-c)^k+b\right]=\sum_{n=0}^{\infty}c_n(x-c)^{kn}$ *for* $|x-c|<R^{1/k}$.

The proofs are left as an exercise.

We close this section with a set of examples.

Example 9.4.8 *Find the power series expansion for* $f(x)=\dfrac{1}{1-x^2}$ *about the point* $\alpha=\dfrac{1}{2}$*and give the radius of convergence.*

Note that

$$\begin{aligned}f(x)=\frac{1}{(1+x)(1-x)}&=\frac{1}{2}\left[\frac{1}{(1-x)}+\frac{1}{(1+x)}\right]\\&=\frac{1}{2}\left[\frac{1}{\left(\frac{1}{2}-\left(x-\frac{1}{2}\right)\right)}+\frac{1}{\left(\frac{3}{2}+\left(x-\frac{1}{2}\right)\right)}\right]\\&=\frac{1}{\left(1-2\left(x-\frac{1}{2}\right)\right)}+\frac{1}{3}\frac{1}{\left(1+\frac{2}{3}\left(x-\frac{1}{2}\right)\right)}.\end{aligned}$$

Since $\dfrac{1}{\left(1-2\left(x-\frac{1}{2}\right)\right)}=\sum_{n=0}^{\infty}2^n\left(x-\frac{1}{2}\right)^n$ *for* $\left|2\left(x-\frac{1}{2}\right)\right|<1$ *or* $\left|x-\frac{1}{2}\right|<\frac{1}{2}$*and* $\dfrac{1}{\left(1+\frac{2}{3}\left(x-\frac{1}{2}\right)\right)}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^n\left(x-\frac{1}{2}\right)^n$ *for* $\left|\frac{2}{3}\left(x-\frac{1}{2}\right)\right|<1$ *or* $\left|x-\frac{1}{2}\right|<\frac{3}{2}$. *Because both series expansions are valid in* $\left|x-\frac{1}{2}\right|<\frac{1}{2}$, *it follows that*

$$f(x)=\sum_{n=0}^{\infty}\left(2^n+\frac{(-2)^n}{3^{n+1}}\right)\left(x-\frac{1}{2}\right)^n \text{ for } \left|x-\frac{1}{2}\right|<\frac{1}{2}.$$

Example 9.4.9 *Find the power series expansion for* $g(x) = \arcsin(x)$ *about the point* $\alpha = 0$.

We know that, for $|x| < 1$, $\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$. *From the Binomial Series Theorem, for* $m = -\frac{1}{2}$, *we have that*

$$(1+u)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} u^n$$ *for* $|u| < 1$. *Since* $|u| < 1$ *if and only if* $\left|u^2\right| < 1$, *it follows that*

$$\left(1-t^2\right)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} (-1)^n t^{2n} \text{ for } |t| < 1.$$

Note that

$$\underbrace{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-n+1\right)}_{n\,terms}(-1)^n = \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+(n-1)\right)$$
$$= \frac{1\cdot 3\cdots(2n-1)}{2^n}.$$

Consequently,

$$\left(1-t^2\right)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2^n n!} t^{2n} \text{ for } |t| < 1$$

with the convergence being uniform in each $|t| \le h$ *for any* h *such that* $0 < h < 1$. *Applying the Integration of Power Series Theorem (Theorem 9.1.13), it follows that*

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{(2n+1)\,2^n n!} x^{2n+1}, \text{ for } |x| < 1$$

where $\arcsin 0 = 0$.

Excursion 9.4.10 *Find the power series expansion about* $\alpha = 0$ *for* $f(x) =$

$\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ *and give the radius of convergence.*

Upon noting that $f(0) = 1$, $f'(0) = 0$, $f^{(2n)}(x) = f(x)$ and $f^{(2n-1)}(x) = f'(x)$, it follows that we can write f as $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ for all $x \in \mathbb{R}$.

Example 9.4.11 *Suppose that we want the power series expansion for* $f(x) = \ln(\cos(x))$ *about the point* $\alpha = 0$. *Find the Taylor Remainder* R_3 *in both the Lagrange and Integral forms.*

Since the Lagrange form for R_3 *is given by* $\dfrac{f^{(4)}(\xi)}{4!}x^4$ *for* $0 < \xi < x$, *we have that*

$$R_3 = \frac{-\left(4\sec^2\xi\tan^2\xi + 2\sec^4\xi\right)x^4}{24} \text{ for } 0 < \xi < x.$$

In general, the integral form is given by $R_n(\alpha, x) = \displaystyle\int_\alpha^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt$. *For this problem,* $\alpha = 0$ *and* $n = 3$, *which gives*

$$R_3(\alpha, x) = \int_\alpha^x \frac{-(x-t)^3}{6}\left(4\sec^2 t\tan^2 t + 2\sec^4 t\right) dt$$

Excursion 9.4.12 *Fill in what is missing in the following application of the geometric series expansion and the theorem on the differentiation of power series to find* $\sum_{n=1}^{\infty} \dfrac{3n-1}{4^n}$.

Because $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ *for* $|x| < 1$*, it follows that*

$$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \underset{(1)}{\underline{\hspace{6cm}}}.$$

From the theorem on differentiation of power series,

$$\sum_{n=1}^{\infty} n x^n = x \left(\frac{x}{1-x}\right)' = \underset{(2)}{\underline{\hspace{6cm}}}$$

in $|x| < 1$*. Hence,*

$$3\sum_{n=1}^{\infty} \frac{n}{4^n} = \underset{(3)}{\underline{\hspace{6cm}}}.$$

Combining the results yields that

$$\sum_{n=1}^{\infty} \frac{3n-1}{4^n} = \sum_{n=1}^{\infty} \left(3\frac{n}{4^n} - \frac{1}{4^n}\right) = \underset{(4)}{\underline{\hspace{4.5cm}}}.$$

Expected responses are: (1) $\frac{1}{3}$, (2) $x\,(1-x)^{-2}$, (3) $\frac{4}{3}$, and (4) 1.

9.5 Fourier Series

Our power series expansions are only useful in terms of representing functions that are nice enough to be continuously differentiable, infinitely often. We would like to be able to have series expansions that represent functions that are not so nicely behaved. In order to obtain series expansions of functions for which we may have only a finite number of derivatives at some points and/or discontinuities at other points, we have to abandon the power series form and seek other "generators." The set of generating functions that lead to what is known as Fourier series is $\{1\} \cup \{\cos nx : n \in \mathbb{J}\} \cup \{\sin nx : n \in \mathbb{J}\}$.

Definition 9.5.1 *A* ***trigonometric series*** *is defined to be a series that can be written in the form*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{9.22}$$

where $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are sequences of constants.*

Definition 9.5.2 *A* ***trigonometric polynomial*** *is a finite sum in the form*

$$\sum_{k=-N}^{N} c_k e^{ikx},\ x \in \mathbb{R} \tag{9.23}$$

where c_k, $k = -N, -N+1, ..., N-1, N$, is a finite sequence of constants.

Remark 9.5.3 *The trigonometric polynomial given in (9.23) is real if and only if $c_{-n} = \overline{c_n}$ for $n = 0, 1, ..., N$.*

Remark 9.5.4 *It follows from equation (9.12) that the Nth partial sum of the trigonometric series given in (9.22) can be written in the form given in (9.23). Consequently, a sum in the form $\frac{1}{2}a_0 + \sum_{k=0}^{N} (a_k \cos kx + b_k \sin kx)$ is also called a trigonometric polynomial. The form used is often a matter of convenience.*

The following "orthogonality relations" are sometimes proved in elementary calculus courses as applications of some methods of integration:

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} \pi & \text{, if} \quad m = n \\ 0 & \text{, if} \quad m \neq n \end{cases}$$

and

$$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0 \text{ for all } m, n \in \mathbb{J}.$$

We will make use of these relations in order to find useful expressions for the coefficients of trigonometric series that are associated with specific functions.

Theorem 9.5.5 *If f is a continuous function on $I = [-\pi, \pi]$ and the trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ converges uniformly to f on I, then*

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\, dt \text{ for } n \in \mathbb{J} \cup \{0\} \tag{9.24}$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\, dt. \tag{9.25}$$

Proof. For each $k \in \mathbb{J}$, let $s_k(x) = \frac{1}{2}a_0 + \sum_{m=1}^{k}(a_m \cos mx + b_m \sin mx)$ and suppose that $\varepsilon > 0$ is given. Because $s_k \underset{I}{\rightrightarrows} f$ there exists a positive integer M such that $k > M$ implies that $|s_k(x) - f(x)| < \varepsilon$ for all $x \in I$. It follows that, for each fixed $n \in \mathbb{J}$,

$$|s_k(x)\cos nx - f(x)\cos nx| = |s_k(x) - f(x)|\,|\cos nx| \le |s_k(x) - f(x)| < \varepsilon$$

and

$$|s_k(x)\sin nx - f(x)\sin nx| = |s_k(x) - f(x)|\,|\sin nx| \le |s_k(x) - f(x)| < \varepsilon$$

for all $x \in I$ and all $k > M$. Therefore, $s_k(x)\cos nx \underset{I}{\rightrightarrows} f(x)\cos nx$ and $s_k(x)\sin nx \underset{I}{\rightrightarrows} f(x)\sin nx$ for each fixed n. Then for fixed $n \in \mathbb{J}$,

$$f(x)\cos nx = \frac{1}{2}a_0\cos nx + \sum_{m=1}^{\infty}(a_m \cos mx \cos nx + b_m \sin mx \cos nx)$$

and

$$f(x)\sin nx = \frac{1}{2}a_0\sin nx + \sum_{m=1}^{\infty}(a_m \cos mx \sin nx + b_m \sin mx \sin nx);$$

the uniform convergence allows for term-by-term integration over the interval $[-\pi, \pi]$ which, from the orthogonality relations yields that

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \pi a_n \quad \text{and} \quad \int_{-\pi}^{\pi} f(x)\sin nx\,dx = \pi b_n.$$

■

Definition 9.5.6 *If f is a continuous function on $I = [-\pi, \pi]$ and the trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges uniformly to f on I, then the trigonometric series*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is called the ***Fourier series*** *for the function f and the numbers a_n and b_n are called the* ***Fourier coefficients*** *of f.*

Given any Riemann integrable function on an interval $[-\pi, \pi]$, we can use the formulas given by (9.24) and (9.25) to calculate Fourier coefficients that could be associated with the function. However, the Fourier series formed using those coefficients may not converge to f. Consequently, a major concern in the study of Fourier series is isolating or describing families of functions for which the associated Fourier series can be identified with the "generating functions"; i.e., we would like to find classes of functions for which each Fourier series generated by a function in the class converges to the generating function.

The discussion of Fourier series in our text highlights some of the convergence properties of Fourier series and the estimating properties of trigonometric polynomials. The following is a theorem that offers a condition under which we have pointwise convergence of the associated Fourier polynomials to the function. The proof can be found on pages 189-190 of our text.

Theorem 9.5.7 *For f a periodic function with period 2π that is Riemann integrable on $[-\pi, \pi]$, let*

$$s_N(f; x) = \sum_{m=-N}^{N} c_m e^{imx} \text{ where } c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{imt} dt.$$

If, for some x, there are constants $\delta > 0$ and $M < \infty$ such that

$$|f(x+t) - f(x)| \le M|t|$$

for all $t \in (-\delta, \delta)$, then $\lim_{N\to\infty} s_N(f; x) = f(x)$.

The following theorem that is offered on page 190 of our text can be thought of as a trigonometric polynomial analog to Taylor's Theorem with Remainder.

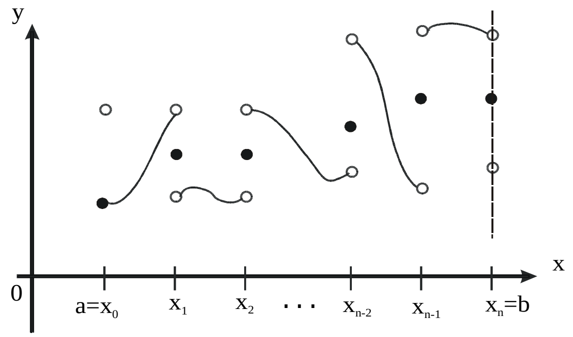
Theorem 9.5.8 *If f is a continuous function that is periodic with period 2π and $\varepsilon > 0$, then there exists a trigonometric polynomial P such that $|P(x) - f(x)| < \varepsilon$ for all $x \in \mathbb{R}$.*

For the remainder of this section, we will focus briefly on the process of finding Fourier series for a specific type of functions.

Definition 9.5.9 *A function f defined on an interval $I = [a, b]$ is* ***piecewise continuous*** *on I if and only if there exists a partition of I, $\{a = x_0, x_1, ..., x_{n-1}, x_n = b\}$ such that (i) f is continuous on each segment (x_{k-1}, x_k) and (ii) $f(a+)$, $f(b-)$ and, for each $k \in \{1, 2, ..., n-1\}$ both $f(x_k+)$ and $f(x_k-)$ exist.*

Definition 9.5.10 *If f is piecewise continuous on an interval I and $x_k \in I$ is a point of discontinuity, then $f(x_k+) - f(x_k-)$ is called the* ***jump at*** *x_k. A piecewise continuous function on an interval I is said to be* ***standardized*** *if the values at points of discontinuity are given by $f(x_k) = \frac{1}{2}\left[f(x_k+) + f(x_k-)\right]$.*

Note that two piecewise continuous functions that differ only at a finite number of points will generate the same associated Fourier coefficients. The following figure illustrates a standardized piecewise continuous function.



Definition 9.5.11 *A function f is* ***piecewise smooth*** *on an interval $I = [a, b]$ if and only if (i) f is piecewise continuous on I, and (ii) f' both exists and is piecewise continuous on the segments corresponding to where f is continuous. The function f is* ***smooth*** *on I if and only if f and f' are continuous on I.*

Definition 9.5.12 *Let f be a piecewise continuous function on $I = [-\pi, \pi]$. Then the* ***periodic extension*** *$\tilde{f}$ of f is defined by*

$$\tilde{f}(x) = \begin{cases} f(x) & , \text{if} \quad -\pi \le x < \pi \\ \dfrac{f(-\pi+) + f(\pi-)}{2} & , \text{if} \quad x = \pi \vee x = -\pi \\ \tilde{f}(x - 2\pi) & , \text{if} \quad x \in \mathbb{R} \end{cases} ,$$

where f is continuous and by $\tilde{f}(x) = \dfrac{f(x+) + f(x-)}{2}$ an each point of discontinuity of f in $(-\pi, \pi)$.

It can be shown that, if f is periodic with period 2π and piecewise smooth on $[-\pi, \pi]$, then the Fourier series of f converges for every real number x to the limit $\frac{f(x+)+f(x-)}{2}$. In particular, the series converges to the value of the given function f at every point of continuity and to the standardized value at each point of discontinuity.

Example 9.5.13 *Let* $f(x) = x$ *on* $I = [-\pi, \pi]$. *Then, for each* $j \in \mathbb{Z}$, *the periodic extension* $\tilde{f}$ *satisfies* $\tilde{f}(j\pi) = 0$ *and the graph in each segment of the form* $(j\pi, (j+1)\pi)$ *is identical to the graph in* $(-\pi, \pi)$. *Use the space provided to sketch a graph for* f.

The associated Fourier coefficients for f *are given by (9.24) and (9.25) from Theorem 9.5.5. Because* $t\cos nt$ *is an odd function,*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt\,dt = 0 \textit{ for } n \in \mathbb{J}\cup\{0\}.$$

According to the formula for integration-by parts, if $n \in \mathbb{J}$, *then*

$$\int t\sin nt\,dt = -\frac{t\cos nt}{n} + \frac{1}{n}\int \cos nt\,dt + C$$

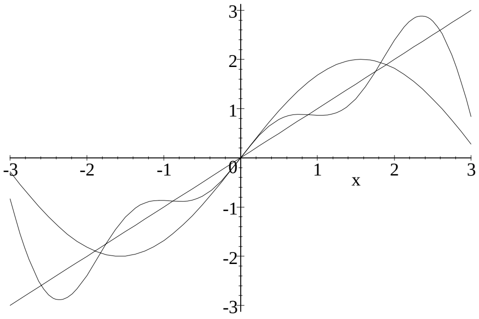
for any constant C. *Hence,* $\cos n\pi = (-1)^n$ *for* $n \in \mathbb{J}$ *yields that*

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt\,dt = \frac{2}{\pi}\int_{0}^{\pi} t\sin nt\,dt = \begin{cases} \frac{2}{n} & , \textit{if} \quad 2 \nmid n \\ -\frac{2}{n} & , \textit{if} \quad 2 \mid n \end{cases}.$$

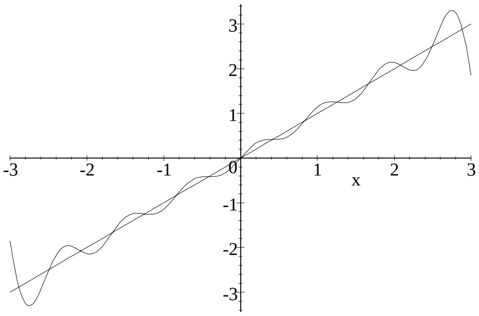
Thus, the Fourier series for f *is given by*

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$$

The following figure shows the graphs of f, $s_1(x) = 2\sin x$, *and* $s_3(x) = 2\sin x - \sin 2x + \dfrac{2}{3}\sin 3x$ *in* $(-3, 3)$.



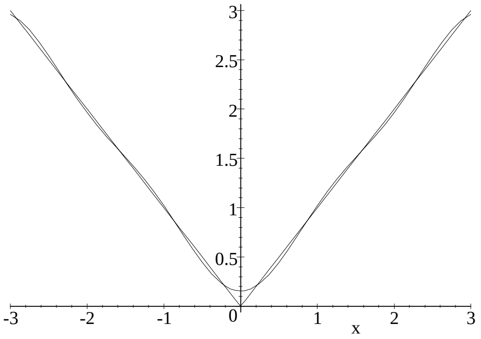
while the following shows the graphs of f *and* $s_7(x) = 2\sum_{n=1}^{7}(-1)^{n+1}\dfrac{\sin nx}{n}$ *in* $(-3, 3)$.



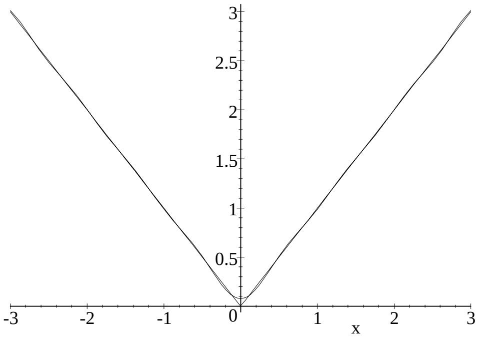
Example 9.5.14 *Find the Fourier series for* $f(x) = |x|$ *in* $-\pi \le x \le \pi$. *Note that, because* f *is an even function,* $f(t)\sin nt$ *is odd.*

Hopefully, you noticed that $b_n = 0$ for each $n \in \mathbb{J}$ and $a_n = 0$ for each even natural number n. Furthermore, $a_0 = \pi$ while, integration-by-parts yielded that $a_n = -4n^{-2}(\pi)^{-1}$ for n odd.

The following figure shows $f(x) = |x|$ and the corresponding Fourier polynomial $s_3(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{9}\cos 3x\right]$ in $(-3, 3)$.



We close with a figure that shows $f(x) = |x|$ and the corresponding Fourier polynomial $s_7(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{7}\frac{1}{(2n-1)^2}\cos(2n-1)x$ in $(-3, 3)$. Note how the difference is almost invisible to the naked eye.



9.6 Problem Set I

1. Apply the Geometric Series Expansion Theorem to find the power series expansion of $f(x) = \dfrac{3}{4-5x}$ about $\alpha = 2$ and justify where the expansion is valid. Then verify that the coefficients obtained satisfy the equation given in part (c) of Theorem 9.1.13.

2. Let

$$g(x) = \begin{cases} \exp\left(-1/x^2\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

   where $\exp w = e^w$.

   (a) Use the Principle of Mathematical Induction to prove that, for each $n \in \mathbb{J}$ and $x \in \mathbb{R} - \{0\}$, $g^{(n)}(x) = x^{-3n} P_n(x) \exp\left(-1/x^2\right)$ where $P_n(x)$ is a polynomial.

   (b) Use l'Hôpital's Rule to justify that, for each $n \in \mathbb{J} \cup \{0\}$, $g^{(n)}(0) = 0$.

3. Use the Ratio Test, as stated in these Companion Notes, to prove Lemma 9.1.8.

4. For each of the following use either the Root Test or the Ratio Test to find the "interval of convergence."

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(7x)^n}{n!}$

   (b) $\displaystyle\sum_{n=0}^{\infty} 3n(x-1)^n$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{\sqrt[n]{n}}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2 (x-3)^n}{(2n)!}$

   (e) $\displaystyle\sum_{n=0}^{\infty} \frac{(\ln n)\, 3^n (x+1)^n}{5^n n\sqrt{n}}$

5. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{\ln(n+1)\, 2^n (x+1)^n}{n+1}$ is convergent in $\left(-\frac{3}{2}, -\frac{1}{2}\right)$.

6. For each of the following, derive the power series expansion about the point α and indicate where it is valid. Remember to briefly justify your work.

   (a) $g(x) = \dfrac{3x-1}{3x+2}$; $\alpha = 1$
   (b) $h(x) = \ln x$; $\alpha = 2$

7. For each of the following, find the power series expansion about $\alpha = 0$.

   (a) $f(x) = \left(1 - x^2\right)^{-1/2}$
   (b) $f(x) = (1-x)^{-2}$
   (c) $f(x) = (1-x)^{-3}$
   (d) $f(x) = \arctan\left(x^2\right)$

8. Find the power series expansion for $h(x) = \ln\left(x + \sqrt{1+x^2}\right)$ about $\alpha = 0$ and its interval of convergence. (Hint: Consider h'.)

9. Prove that if $f(u) = \sum_{n=0}^{\infty} c_n (u-b)^n$ for $|u-b| < R$ with $R > 0$ and $b = kc + d$ with $k \neq 0$, then $f(kx+d) = \sum_{n=0}^{\infty} c_n k^n (x-c)^n$ for $|x-c| < \dfrac{R}{|k|}$.

10. Prove that if $f(u) = \sum_{n=0}^{\infty} c_n (u-b)^n$ for $|u-b| < R$ with $R > 0$, then $f\left[(x-c)^k + b\right] = \sum_{n=0}^{\infty} c_n (x-c)^{kn}$ in $|x-c| < R^{1/k}$ for any fixed positive integer k.

11. Find the power series expansions for each of the following about the specified point α.

    (a) $f(x) = (3x+5)^{-2}$; $\alpha = 1$
    (b) $g(x) = \sin x \cos x$; $\alpha = \dfrac{\pi}{4}$
    (c) $h(x) = \ln\left(\dfrac{x}{(1-x)^2}\right)$; $\alpha = 2$

12. Starting from the geometric series $\sum_{n=1}^{\infty} x^n = x(1-x)^{-1}$ for $|x| < 1$, derive closed form expressions for each of the following.

    (a) $\sum_{n=1}^{\infty}(n+1)x^n$

    (b) $\sum_{n=1}^{\infty}(n+1)x^{2n}$

    (c) $\sum_{n=1}^{\infty}(n+1)x^{n+2}$

    (d) $\sum_{n=1}^{\infty}\frac{n+1}{n+3}x^{n+3}$

13. Find each of the following, justifying your work carefully.

    (a) $\sum_{n=1}^{\infty}\frac{n^2+2n-1}{3^n}$

    (b) $\sum_{n=1}^{\infty}\frac{n(3^n-2^n)}{6^n}$

14. Verify the orthogonality relations that were stated in the last section.

    (a) $\int_{-\pi}^{\pi}\cos mx\cos nx\,dx = \int_{-\pi}^{\pi}\sin mx\sin nx\,dx = \begin{cases}\pi & \text{, if } m=n\\ 0 & \text{, if } m\neq n\end{cases}.$

    (b) $\int_{-\pi}^{\pi}\cos mx\sin nx\,dx = 0$ for all $m, n \in \mathbb{J}$.

15. For each of the following, verify that the given Fourier series is the one associated with the function f according to Theorem 9.5.5.

    (a) $f(x) = \begin{cases} 0 & \text{, if } -\pi \le x < 0\\ 1 & \text{, if } 0 \le x \le \pi\end{cases}$; $\frac{1}{2}+\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)x)}{2k+1}$

(b) $f\,(x) = x^2$ for $x \in [-\pi, \pi]$; $\dfrac{\pi^2}{3} + 4\displaystyle\sum_{k=1}^{\infty} (-1)^k \dfrac{\cos(kx)}{k^2}$

(c) $f\,(x) = \sin^2 x$ for $x \in [-\pi, \pi]$; $\dfrac{1}{2} - \dfrac{\cos 2x}{2}$

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